

Light-Front Realization of Chiral Symmetry Breaking

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We discuss a description of chiral symmetry breaking in the light-front (LF) formalism. Based on careful analyses of several models, we give clear answers to the following three fundamental questions: (i) What is the difference between the LF chiral transformation and the ordinary chiral transformation? (ii) How does a gap equation for the chiral condensate emerge? (iii) What is the consequence of the coexistence of a nonzero chiral condensate and the trivial Fock vacuum? The answer to Question (i) is given through a classical analysis of each model. Question (ii) is answered based on our recognition of the importance of characteristic constraints, such as the zero-mode and fermionic constraints. Question (iii) is intimately related to another important problem, reconciliation of the nonzero chiral condensate $\langle \bar{\Psi} \Psi \rangle \neq 0$ and the invariance of the vacuum under the LF chiral transformation $Q_5^{\text{LF}} |0\rangle = 0$. This and Question (iii) are understood in terms of the modified chiral transformation laws of the dependent variables. The characteristic ways in which the chiral symmetry breaking is realized are that the chiral charge Q_5^{LF} is no longer conserved and that the transformation of the scalar and pseudoscalar fields is modified. We also discuss other outcomes, such as the light-cone wave function of the pseudoscalar meson in the Nambu-Jona-Lasinio model.

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I. INTRODUCTION

Dirac's first proposal of the "front form," which is now called the light-front (LF) Hamiltonian formalism, intended to combine special relativity and quantum mechanics. [1] Since the quantum mechanics is based on Hamiltonian formalism, Dirac's implied intention was to find the most convenient "form" for relativistic quantum systems. In the usual instant form, with the equal-time hypersurface $x^0 = \text{const}$,¹ translations and rotations transform quantum states in a very simple manner, because they do not change the quantization surface. These kinds of static Poincaré generators are called "kinematical". The other generators, which change the quantization surface, are called "dynamical". For example, the Lorentz boost mixes time and space coordinates and thus changes the hypersurface. The Hamiltonian itself, of course, is a dynamical operator. Therefore, an eigenstate in a rest frame is no longer an eigenstate in the boosted frame. Even though we know eigenstates in the rest frame, to find a new eigenstate in the boosted frame requires, in principle, as much as effort as solving the entire problem. Contrastingly, the front form in which we treat $x^+ = (x^0 + x^3)/\sqrt{2}$ as time, allows the maximum number of kinematical operators in ten Poincaré generators. In particular, the boost operator now forms a part of the *kinematical* operators. With the LF coordinates,² the boost transformation along the third axis is simply given by $x^\pm \rightarrow e^{\pm\phi}x^\pm$ ($\tanh\phi = \beta$), and it does not change the quantization surface $x^+ = 0$. Due to this simplicity, it is quite easy to construct a boosted eigenstate $|n; P'\rangle$ from $|n; P\rangle$, where $P'^\mu = \Lambda^\mu_\nu P^\nu$. Therefore, the front form is considered to be an ideal framework for the description of relativistic quantum systems.

This idea is convenient for understanding the hadron physics based on the quantum chromodynamics (QCD), because all the hadrons are relativistic quantum bound states of quarks and gluons. Of course there exists a covariant formalism of quantum field theories, but at present its success is, strictly speaking, limited only to the perturbative region, and thus effort should be made to consider other possible frameworks for relativistic quantum systems. One of the merits of using Hamiltonian formalism is that we can utilize various nonperturbative techniques (such as the variational approximation, the Tamm-Danoff approximation, etc.) developed in quantum mechanics. Moreover, in some aspects, the LF field theory has a structure similar to that of the nonrelativistic theory. This resemblance also allows us to use familiar nonperturbative methods of nonrelativistic quantum mechanics. The other merits of the Hamiltonian formalism are that we can compute both the wave functions and eigenvalues simultaneously, and that it can in principle describe the time (x^+) evolution of the system. The accessibility to such dynamical information deserves more attention and investigation.

Dirac's LF Hamiltonian formalism has been applied to various quantum field theories. [2] The most significant success was made in 1+1 dimensional QCD, where the mass spectra and wavefunctions of various mesons and baryons for various numbers of colors were calculated by explicit diagonalization of the LF Hamiltonian. [3] An interesting application of the LF formalism to time dependence of a relativistic system was made for the scattering of two composite particles. [4]

Many of the properties of the LF formalism are related to the unique structure of the dispersion relation

$$p^- = \frac{p_\perp^2 + m^2}{2p^+}, \quad (1.1)$$

where $p^\mp = (p^0 \mp p^3)/\sqrt{2}$ are the LF energy ($-$) and the longitudinal momentum ($+$). Unlike the usual instant form dispersion $E = \pm\sqrt{\mathbf{p}^2 + m^2}$, the LF dispersion relation has a rational structure.³ This leads to an important property, "vacuum simplicity". Requiring the semi-positivity of the LF energy, $p^- \geq 0$, we immediately find a restriction on the longitudinal momentum, $p^+ \geq 0$. Therefore, no particle state with nonzero momentum can mix with the Fock vacuum $|0\rangle$, which has zero momentum. (We cannot construct a zero momentum from positive momenta.) This fact implies that the structure of the vacuum is very simple. Note also that the large value of the LF energy involves two different cases, that of the small p^+ limit and that of the large p_\perp limit. If we set the infrared cutoff as $\epsilon < p^+$ to regularize the divergent energy, the subtlety of the $p^+ = 0$ case is removed, and the vacuum of the system becomes the trivial Fock vacuum. This is one of the reasons why some people use the somewhat exaggerated term "vacuum triviality".

The LF vacuum also simplifies the description of excited states. Roughly speaking, this is because any Fock state should be directly related to the excited states. Indeed, in a method called the discretized light-cone quantization

¹In what follows, we simply refer to this standard scheme as the equal-time (ET) quantization.

²Our definition of the light-front variables and some conventions are given in Appendix A.

³If we regard p^+ as mass, Eq. (1.1) can be understood as a nonrelativistic dispersion in (transverse) 2 dimensions, $E_\perp = p_\perp^2/2m_\perp + E_0$.

(DLCQ) method, [7,8] which is discussed below, the dimension of the Hamiltonian matrix in the longitudinal direction is finite in the calculation, and the valence description is a good approximation for excited states. This situation is very advantageous for QCD, since it suggests the constituent picture, which allows us an intuitive understanding of low energy hadronic properties. In fact, one of the strong motivations which drives the present resurgence of the LF field theory is our anticipation that it might connect the constituent quark model and QCD at a field theoretic level. [5]

However, we must be careful not to be too optimistic about such a naive expectation: The constituent picture is not just a valence picture. The constituent picture should emerge as a result of a significant nonperturbative phenomenon, the dynamical chiral symmetry breaking (D χ SB). [6] According to the conventional formulation, D χ SB is thought to be described by determining a new vacuum state that breaks the chiral symmetry but is energetically favored. A condition that minimizes the vacuum energy is the so-called gap equation, which is generally a nonlinear equation for the order parameter of interest. On the other hand, the structure of the LF vacuum is kinematically determined to be simple and seems independent of the dynamics. So a natural question arises: *How can we reconcile a LF “trivial” vacuum with a chirally broken ET vacuum having a nonzero fermion condensate?* Actually this problem is not restricted to D χ SB. Most nonperturbative phenomena are currently believed to be related to some nontrivial structure of the vacuum. In addition to D χ SB, such “vacuum physics” may include topological effects (theta vacua), gluon condensation, the Higgs mechanism, and even confinement. Therefore, we should ask more generally the following: *How can we represent “vacuum physics” in the LF field theories?* The primary purpose of the present paper is to determine a way of describing D χ SB on the LF. Such consideration is necessary for realizing the constituent picture, but we hope that it will also provide useful information for the understanding of other types of vacuum physics.

We wish to answer the above questions, and we now set out to do so. It is now widely accepted that vacuum physics will be described on the LF as *physics of the longitudinal zero mode* (simply referred to as the “zero mode”), which is the only point with respect to which the vacuum triviality becomes subtle. In other words, if the LF formalism is a correct framework, all the information concerning vacuum physics should be extractable from the vicinity of the zero mode. This is a very delicate problem involving the infrared (IR) singularity resulting from an infinitesimally small longitudinal momentum. Therefore, the most important point when we discuss vacuum physics on the LF is determining how to regularize the IR singularity. Generally speaking, three different methods have been proposed for the study of such “zero-mode physics”. The relation among them is not clear at present time. We discuss these methods below.

1. The DLCQ method [7,8]

The most straightforward but powerful method is the DLCQ method, which treats the zero mode explicitly by compactifying the longitudinal space into a circle $x^- \in [-L, L]$ with appropriate boundary conditions on fields. [7] All the longitudinal momenta take discrete values. For a scalar field, we impose the periodic boundary condition at each LF time $\phi(x^- = -L, x_\perp) = \phi(x^- = L, x_\perp)$, so that we can unambiguously define the longitudinal zero mode:

$$\phi_0(x_\perp) = \frac{1}{2L} \int_{-L}^L dx^- \phi(x). \quad (1.2)$$

Then the scalar field is decomposed into the zero mode and the remaining oscillation modes: $\phi(x) = \phi_0(x_\perp) + \varphi(x)$. Since the vacuum expectation value of φ is zero, the central issue is determining how to evaluate the zero-mode part $\langle 0 | \phi_0 | 0 \rangle$.

There is another important merit in this approach. [8] If we confine ourselves to the particle modes, then the positivity and discreteness of the longitudinal momenta lead to a finite-dimensional Hamiltonian:⁴ The number of partitions of the total momenta P^+ into positive momenta $P^+ = \sum_n p_n^+$ is now finite. Thus, it is easy to diagonalize the Hamiltonian numerically, which gives the eigenvalues and wavefunctions of excited states simultaneously.

All the calculations are done with finite L after that we take the infinite volume limit $L \rightarrow \infty$.

2. Setting infrared cutoffs for the longitudinal momenta

The secondary method consists of eliminating the zero mode from the theory. As we suggested above, if we

⁴Strictly speaking, this holds only in 1+1 dimensions. We do not have any restriction to transverse directions. Nevertheless, the situation is much simpler than that in the ET quantization.

set an infrared cutoff for the longitudinal momentum as $p^+ > \epsilon$, then we have no zero mode, and the vacuum becomes trivial. In this case, the information concerning vacuum physics must be carried by counterterms, which remove any infrared divergence. If this is the case, we must perform “nonperturbative” renormalization in the Hamiltonian formalism. This is actually a very difficult task, due to the lack of the Lorentz covariance. It leads to a disastrous situation, with infinitely many counterterms. [5] To successfully treat them is a challenging problem, and some ideas (such as coupling coherence in the similarity renormalization group method) have been developed, but we do not discuss these in the present paper. This is simply because one of our aims is to describe D χ SB in the Nambu–Jona-Lasinio (NJL) model with the non-renormalizable property. As long as we consider the NJL model, we do not have to regard D χ SB as a problem of renormalization, though a regularization scheme is a great matter in the NJL model.

3. Working off the light cone

In order to regularize the zero-mode singularity, we can work slightly off the LF and define the LF theory as a limit of the theory so obtained. [9] This can be achieved by choosing near-light-front coordinates like $x^+ = x^0 \sin \theta + x^3 \cos \theta$ and $x^- = x^0 \cos \theta - x^3 \sin \theta$ with $\theta \sim \pi/4$. However, this method lacks the primary merit of the LF formalism, the simplicity of the vacuum, and thus the amount of calculations it requires is almost the same as, or greater than, that for the usual ET calculation. This approach is pedagogical and suggestive, but it is not a LF theory, and simply for this reason, we do not consider it here even though we learn much from it.

We have been attacking the problem of describing vacuum physics on the LF from the viewpoint of zero-mode physics. [10–17] In the present paper, based on our previous work, we focus on the chiral symmetry breaking [10–14] and the effects of coupling to the gauge field. [16,17] We follow the DLCQ method when we have scalar fields. Considerable efforts have been made with this method [18] for the spontaneous (discrete) symmetry breaking in a simple scalar model ($\lambda \phi_{1+1}^4$). In these works, results have been obtained that are consistent with the conventional results for the critical coupling as well as the nonzero vacuum expectation value $\langle 0 | \phi | 0 \rangle \neq 0$. We see below that this method is also applicable to models with chiral symmetry. When we have only fermionic fields, we do not use any of the three methods described above. However, we carefully consider the infrared singularities associated with the longitudinal zero mode of the fermion bilinear. In this sense, our approach in this case is physically similar to the DLCQ method.

Finally, let us comment on another important issue. When we discuss the chiral symmetry on the LF, we must be careful about its special properties. On the LF, the chiral transformation is defined differently from the ordinary one. The reason for this is the following. Half of the degrees of freedom of any LF fermion are dependent variables to be represented by other independent variables. Thus, it is natural to define the chiral transformation only for the independent component of the fermion. As we will discuss, it is not evident whether the LF chiral transformation thus defined is equivalent to the ordinary one. In fact, most surprisingly, it is not equivalent in the simplest case, as a massive free fermion theory is invariant under the LF chiral transformation. [19] We discuss this problem in great detail.

With all these points in mind, we can rephrase the above questions in more concrete terms as follows:

- (i) *What is the difference between the LF chiral transformation and the ordinary one?* Even though we are ultimately interested in the breaking of the usual chiral symmetry, we must know the differences and similarities of the two. This should be done both at classical and quantum levels.
- (ii) *How does a gap equation for the chiral condensate emerge?* Quite generally, we need to resort to a gap equation in finding a nonzero value of the order parameter. This should be the case even in the LF formalism.
- (iii) *What is the consequence of the coexistence of a nonzero chiral condensate and the trivial Fock vacuum?* We must understand why it is possible for the Fock vacuum to give a nonzero condensate and how the theory should be modified.

We give clear answers to these questions based on the study of several models. First, in the next section, we highlight the above-mentioned unusual aspects of the LF chiral transformation. Using the free field theory as a simple example, we demonstrate the difference between the LF chiral transformation and the usual one. This partially answers Question (i). More realistic consideration with interacting field theories is given in Sect. III. There, we discuss the sigma model, [20] the chiral Yukawa model, [13] and the NJL model [14] as examples of a scalar theory, a scalar and fermionic theory, and a purely fermionic theory, respectively. We also investigate in each model how the gap equations for the chiral condensate appear. This corresponds to the answer to Question (ii). Such a model-dependent study is inevitable to understand the chiral symmetry breaking on the LF. Then, we discuss in Sect. IV common features in the methods of realizing D χ SB on the LF. It turns out that the chiral transformation of the dependent variables are necessarily modified in the symmetry breaking phase. Such modification significantly affects the system and resolves

the somewhat paradoxical situations regarding the trivial vacuum and chiral symmetry breaking, and between the nonzero chiral condensate and the invariance of the vacuum under the LF chiral transformation. This analysis should provide the answer to Question (iii). In Sect. V, we consider the effects of coupling to the gauge field. As an example of gauge theories, we discuss the $U(1)$ -gauged sigma model, which is equivalent to the Abelian Higgs model. [16,17] A summary and conclusions are given in the last section.

II. CHIRAL TRANSFORMATION ON THE LIGHT FRONT

In this section, we describe the basic ingredients of chiral symmetry on the LF. Starting from the definition of the LF chiral transformation, we demonstrate how it is different from the ordinary one by using the free fermion theory. A most remarkable difference is the fact that the massive free fermion theory is invariant under the LF chiral transformation. (However, it turns out that this unusual property is highly exceptional. Indeed, it cannot be seen in any models we discuss.) We also comment on the chiral transformation of the scalar field and on the order parameter.

A. Definition

As we have already commented, the definition of the chiral transformation on the LF is different from the usual one. This is due to the special nature of the canonical structure of the LF fermionic field. To see this, it is sufficient to consider the kinetic term of the Lagrangian $\mathcal{L}_0 = \bar{\Psi} i\partial\Psi$. Splitting the fermion field into the “good” and “bad” components as

$$\Psi = \psi_+ + \psi_-, \quad \psi_{\pm} = \Lambda_{\pm}\Psi \quad (2.1)$$

by using the projectors $\Lambda_{\pm} = \gamma^{\mp}\gamma^{\pm}/2$, we easily find that the “bad” component ψ_- is a dependent variable: $\mathcal{L}_0 = \sqrt{2}\psi_+^\dagger i\partial_+\psi_+ + \sqrt{2}\psi_-^\dagger i\partial_-\psi_- + \dots$. Note that ∂_- is a spatial derivative. The canonically independent variable consists of only the “good” component ψ_+ . It is convenient for practical calculations to use the “two-component representation” of the γ matrices, so that the projectors Λ_{\pm} are expressed as

$$\Lambda_+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \quad (2.2)$$

Then, the projected fermions have only upper or lower components:

$$\psi_+ = \Lambda_+\Psi \equiv 2^{-\frac{1}{4}} \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \quad \psi_- = \Lambda_-\Psi \equiv 2^{-\frac{1}{4}} \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \quad (2.3)$$

where we have defined the two-component spinors ψ and χ . Among various representations satisfying Eq. (2.2), we use the following one in the present paper:

$$\gamma^+ = \sqrt{2} \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^- = \sqrt{2} \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}, \quad (2.4)$$

for $i = 1, 2$ and

$$\gamma_5 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}. \quad (2.5)$$

Now let us define the chiral transformation on the LF. Since the bad component χ is a constrained variable in the LF formalism, it is subject to a constraint equation, what we call the “fermionic constraint”:

$$i\partial_-\chi = \frac{1}{\sqrt{2}} (-\sigma^i\partial_i + m) \psi - \frac{\delta}{\delta\chi^\dagger} \mathcal{L}_{\text{int}}[\psi, \chi, \dots].$$

This is in general a nonlinear equation with respect to χ . Therefore, we cannot impose the chiral transformation freely on the whole field. We must impose it so that it is consistent with the constraint relation, or apply it only on

the good component from the beginning. Since a consistency check is an involved task in interacting theories, we just define the chiral transformation only for the good component:

$$\psi \longrightarrow e^{i\theta\sigma^3} \psi . \quad (2.6)$$

This is the definition of the LF chiral transformation. If we do not use the specific two-component representation, it is written $\psi_+ \longrightarrow e^{i\theta\gamma_5} \psi_+$. The transformation of the bad component can be found if we solve the constraint equation. We indeed do check this using some models in the next section. However, before doing that, it is a good exercise to consider the simplest case, that of a free fermion.

B. Massive free fermion

The most impressive and distinguishing property of the LF chiral transformation is that the free fermion theory is invariant under the transformation (2.6) *even if it has a mass term*. Let us confirm this amazing fact explicitly at the Lagrangian level. First of all, it is convenient to separate the solution of the fermionic constraint $\chi = (\sqrt{2}i\partial_-)^{-1}(-\sigma^i\partial_i + m)\psi$ into mass-independent and mass-dependent parts $\chi = \chi^{(0)} + \chi^{(m)}$ as

$$\chi^{(0)} = -\frac{1}{\sqrt{2}}\sigma^i\partial_i \frac{1}{i\partial_-}\psi , \quad \chi^{(m)} = \frac{m}{\sqrt{2}}\frac{1}{i\partial_-}\psi ,$$

where $(i\partial_-)^{-1}$ is defined in Appendix A. The LF chiral transformation (2.6) induces

$$\chi^{(0)} \longrightarrow e^{-i\theta\sigma^3} \chi^{(0)} , \quad \chi^{(m)} \longrightarrow e^{i\theta\sigma^3} \chi^{(m)} . \quad (2.7)$$

When $m = 0$, we have $\chi^{(m)} = 0$. Thus the chiral transformation of the full field Ψ turns out to be equivalent to the usual one [recall the representation of γ_5 , Eq. (2.5)]:

$$\Psi = 2^{-1/4} \begin{pmatrix} \psi \\ \chi^{(0)} \end{pmatrix} , \quad \Psi \longrightarrow e^{i\theta\gamma_5} \Psi . \quad (2.8)$$

In this case, everything is the same as in the usual chiral transformation.

The situation is different for the massive case. Here, the free fermion Lagrangian is compactly expressed as $\mathcal{L}_{\text{free}} = \bar{\Psi}(i\partial_0 - m)\Psi = \psi^\dagger \omega_{\text{EOM}} + \chi^\dagger \omega_{\text{FC}}$, where $\omega_{\text{EOM}} = i\partial_+\psi - \frac{1}{\sqrt{2}}(\sigma^i\partial_i + m)\chi = 0$ is the equation of motion for ψ and $\omega_{\text{FC}} = i\partial_-\chi - \frac{1}{\sqrt{2}}(-\sigma^i\partial_i + m)\psi = 0$ is the fermionic constraint. The second term here is zero, and it is invariant under the LF chiral transformation. Now, substituting $\chi = \chi^{(0)} + \chi^{(m)}$ into the Lagrangian, we find that the first term $\psi^\dagger \omega_{\text{EOM}}$ is decomposed into obviously invariant and (seemingly) non-invariant terms

$$\psi^\dagger \omega_{\text{EOM}} = \psi^\dagger \left[i\partial_+\psi - \frac{1}{\sqrt{2}} \left(\sigma^i\partial_i\chi^{(0)} + m\chi^{(m)} \right) \right] + \psi^\dagger \left[-\frac{1}{\sqrt{2}} \left(\sigma^i\partial_i\chi^{(m)} + m\chi^{(0)} \right) \right] .$$

The first term of this contribution consists of an m -independent term and a term quadratic in m while the second term depends linearly on m . The apparent invariance of the $\mathcal{O}(m^2)$ term is very intriguing. The $\mathcal{O}(m)$ term changes under the chiral transformation, but thanks to a relation between $\chi^{(0)}$ and $\chi^{(m)}$,

$$\sigma^i\partial_i\chi^{(m)} + m\chi^{(0)} = 0 , \quad (2.9)$$

it eventually vanishes, and therefore the Lagrangian is totally invariant even if we have a mass term. Note that the invariance is closed within each term of $\mathcal{O}(m^n)$, and there is no mixing or cancellation between different order contributions. Now that we have a symmetry, a corresponding conserved Noether current should follow: [19]

$$j_5^\mu = \bar{\Psi}\gamma^\mu\gamma_5\Psi - m\bar{\Psi}\gamma^\mu\gamma_5\frac{1}{i\partial_-}\gamma^+\psi_+ . \quad (2.10)$$

This, of course, reduces to the usual current in the massless limit,

$$J_5^\mu = \bar{\Psi}\gamma^\mu\gamma_5\Psi . \quad (2.11)$$

The chiral charge is given by

$$Q_5^{\text{LF}} = \int_{-\infty}^{\infty} dx^- d^2x_\perp j_5^+(x) = \int_{-\infty}^{\infty} dx^- d^2x_\perp \psi^\dagger \sigma^3 \psi . \quad (2.12)$$

Since $j_5^+ = J_5^+$, the chiral charge is the same as that in the massless case. This is reasonable, because the chiral transformation is defined irrespective of whether we have a mass term.

In a quantum theory, the normal ordered chiral charge is given by

$$: Q_5^{\text{LF}} := \int_0^{\infty} dk^+ \int_{-\infty}^{\infty} d^2k_\perp (b^\dagger(\mathbf{k})\sigma_3 b(\mathbf{k}) + d^\dagger(\mathbf{k})\sigma_3 d(\mathbf{k})) , \quad (2.13)$$

where the mode expansion of the field is done at $x^+ = 0$, as in Ref. [21]:

$$\psi(x) = \int_{-\infty}^{\infty} \frac{d^2k_\perp}{2\pi} \int_0^{\infty} \frac{dk^+}{\sqrt{2\pi}} [b(\mathbf{k})e^{i\mathbf{k}\mathbf{x}} + d^\dagger(\mathbf{k})e^{-i\mathbf{k}\mathbf{x}}] , \quad (2.14)$$

with $\mathbf{k}\mathbf{x} \equiv -k^+x^- + k_\perp^i x_\perp^i$. Note that the quantization has been performed only for the good component ψ_α ($\alpha = 1, 2$ is a spinor index):

$$\left\{ \psi_\alpha(x), \psi_\beta^\dagger(y) \right\}_{x^+ = y^+} = \delta_{\alpha\beta} \delta(x^- - y^-) \delta^{(2)}(x_\perp - y_\perp) . \quad (2.15)$$

If we use the mode expansion with helicity basis, Q_5^{LF} measures the net helicity of a state. Since the chiral charge does not contain a $b^\dagger d^\dagger$ term, it annihilates the vacuum $|0\rangle$ irrespective of the existence of a mass term:

$$: Q_5^{\text{LF}} : |0\rangle = 0 . \quad (2.16)$$

This is a general property of the charge defined on the LF (null-plane charge). Every charge operator is defined as the spatial integration of the $+$ -component of a current operator, which implies that the charge operator has longitudinally zero momentum. Thus the null-plane charge cannot contain any terms like $b^\dagger d^\dagger$ or bd that cannot form zero longitudinal momentum.⁵ Therefore, this charge *always* annihilates the vacuum, irrespective of symmetry. [22] The same argument holds for the Hamiltonian, which is the charge of the energy-momentum tensor. Thus, $: P^- : |0\rangle = 0$ as long as we can ignore the zero modes.

The fact that the chiral current j_5^μ is a conserved current even for $m \neq 0$ is a distinguishable property of the “LF” chiral transformation. However, if we are ultimately interested in the usual chiral symmetry and its breaking, it would be better to treat J_5^μ instead of j_5^μ . Here J_5^μ was defined in Eq. (2.11) as a chiral current for a massless fermion. If we use it even for the massive case, the divergence of J_5^μ gives the ordinary relation

$$\partial_\mu J_5^\mu = 2m\bar{\Psi}i\gamma_5\Psi . \quad (2.17)$$

This can be easily verified by using the Euler-Lagrange equation for a massive fermion. The unexpected symmetry of the massive fermion is certainly a unique and intriguing property of the LF chiral transformation, but it is not clear whether such an unusual situation holds even for general interacting theories. If it is not the case in interacting models, we do not have to consider it. In fact, as we will see in the next section, neither the chiral Yukawa model nor the NJL model possesses this symmetry.

C. Scalar field

Let us comment on the chiral transformation of the scalar fields (σ, π) . If we use the DLCQ method, we find a situation similar to that for the chiral transformation of fermions. Suppose that the system is invariant under the $U(1)$ chiral symmetry:

⁵The fermionic field does not have zero modes because we normally use antisymmetric boundary conditions in the longitudinal direction.

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \rightarrow \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}. \quad (2.18)$$

The same discussion as that given for the fermionic fields can be applied to the case of scalar fields in the DLCQ approach. This is because the zero modes are constrained variables to be determined by nonzero modes (see Sect. IIIA). Hence, it is natural to define the chiral transformation only for the oscillating modes:

$$\begin{pmatrix} \varphi_\sigma \\ \varphi_\pi \end{pmatrix} \rightarrow \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \varphi_\sigma \\ \varphi_\pi \end{pmatrix}, \quad (2.19)$$

where $\sigma(x) = \sigma_0(x_\perp) + \varphi_\sigma(x)$ and $\pi(x) = \pi_0(x_\perp) + \varphi_\pi(x)$. The transformation of the zero modes should follow from the solutions to the constraint relations for the zero modes. More detailed discussion on this point is given in the next section.

D. Order parameters

Usually broken phase is characterized by nonzero vacuum expectation values (VEVs) of the order parameters. For example, the order parameters are the scalar field itself $\langle \sigma \rangle$ for the $U(1)$ symmetry in the sigma model and the fermion bilinear operator $\langle \bar{\Psi} \Psi \rangle$ for the chiral symmetry breaking. In the usual ET formulation, such nonzero values are realized by selecting a nontrivial vacuum that is different from the Fock vacuum. On the other hand, the vacuum in the LF formulation is determined kinematically to be the Fock vacuum. This leads us to ask: How is it possible for those order parameters to take nonzero values in the LF formalism? A hint for the answer to this question lies in the fact that *both order parameters contain dependent variables in the LF formalism*. In the above two examples, they are $\sigma = \sigma_0 + \varphi_\sigma$ and

$$\bar{\Psi} \Psi = \frac{1}{\sqrt{2}}(\psi^\dagger \chi + \chi^\dagger \psi), \quad (2.20)$$

where the zero mode σ_0 and the bad component χ are dependent variables to be determined by the independent variables through constraint relations. Since the constraint equations reflect an interaction, information concerning the dynamics enters the order parameters in terms of these dependent variables. This might lead to nonzero VEVs of the order parameters. We see that this is indeed the case. Note that, without information regarding the interaction, order parameters are not able to have nonzero values. In the ET formalism, this information is of course supplied by the new vacuum. Since the LF vacuum is determined kinematically, it is natural to expect that the dynamical information is supplied by the solution of the constraint. This is the only point with regard to which the information about the interaction can enter the order parameter.

We must be careful about a use of the term “order parameters” in the LF formulation. Its meaning is a little different from the usual one. To see this, recall the reason that $\langle \bar{\Psi} \Psi \rangle$ can be considered as the order parameter in the ET formulation. The reason is clear: The chiral transformation of the fermion bilinears is

$$[Q_5^{\text{ET}}, \bar{\Psi} i \gamma_5 \Psi] = -2i \bar{\Psi} \Psi \quad (2.21)$$

and therefore a nonzero value of $\langle \bar{\Psi} \Psi \rangle$ immediately implies the violation of chiral symmetry. On the other hand, as we saw above, any charge operator in the LF formulation annihilates the vacuum, irrespective of symmetry. Therefore if Eq. (2.21) held in the LF formulation, a nonzero value of $\langle \bar{\Psi} \Psi \rangle$ would seem to be prohibited on the LF. Hence it is not clear whether the quantity $\langle \bar{\Psi} \Psi \rangle$ plays the role of the order parameter in exactly the same way as in the ET quantization. In what follows, for practical purposes, we identify the phase according to the value of $\langle \bar{\Psi} \Psi \rangle$, because our eventual interest is in the breaking of the ordinary chiral symmetry. Hence, if we find $\langle \bar{\Psi} \Psi \rangle \neq 0$ in the LF formalism, we say that the system is in the broken phase. However, we must not forget about the above paradoxical situation. We resolve this problem in a later section.

III. APPEARANCE OF THE GAP EQUATIONS: MODEL ANALYSIS

The main aim of this section is to demonstrate how to obtain the gap equations by using concrete models. Recall that to find nontrivial gap equations is an indispensable step towards the symmetry breaking. This is of course true

of the LF formalism. Here we see the unique way that they appear in the LF formalism. The models we consider here are the sigma model, the chiral Yukawa model, and the NJL model, which are examples of a scalar theory, a coupled system of scalar and fermion fields, and a purely fermionic theory, respectively. What is common among these models is the appearance of the gap equations from characteristic constraint equations of the LF formalism. Various physical consequences, which follow after we obtain nonzero condensates, are discussed in the next section.

A. The sigma model – Scalar fields

The first example we consider is the linear sigma model. [20] Its simplest version, with only scalar fields and a potential with the wrong sign for the mass term, is a basic and classic model of spontaneous symmetry breaking. Actually this “tree-level” symmetry breaking is not of interest presently. We are interested in *dynamical* chiral symmetry breaking. However, we can learn much from the analysis about how we should treat the zero modes. The model is defined by

$$\mathcal{L} = \frac{1}{2} \{(\partial\sigma)^2 + (\partial\pi)^2\} + \frac{1}{2}\mu^2(\sigma^2 + \pi^2) - \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 + c\sigma, \quad (3.1)$$

where the last term is the explicit breaking term, and the potential has the wrong sign for the mass term. In order to treat the longitudinal zero mode appropriately, we use the DLCQ method, in which we compactify the longitudinal space into a circle with periodic boundary conditions on the scalars. Then the zero modes are given as

$$\sigma_0(x_\perp) = \frac{1}{2L} \int_{-L}^L dx^- \sigma(x), \quad \pi_0(x_\perp) = \frac{1}{2L} \int_{-L}^L dx^- \pi(x), \quad (3.2)$$

which leads to a clear separation of the fields $\sigma(x) = \sigma_0(x_\perp) + \varphi_\sigma(x)$ and $\pi(x) = \pi_0(x_\perp) + \varphi_\pi(x)$.

As emphasized above, the most distinguishing feature of a scalar field in the DLCQ method is that the zero mode is not an independent degree of freedom. This is because the momentum conjugate to the longitudinal zero mode drops out of the Lagrangian. (Note that the Lagrangian is linear in terms of the LF time derivative: $\frac{1}{2}(\partial\sigma)^2 = \partial_+\sigma\partial_-\sigma - \frac{1}{2}(\partial_\perp\sigma)^2$.) Thus, the zero modes are subject to the constraint relations

$$\frac{1}{2L} \int_{-L}^L dx^- \{-(\mu^2 + \partial_\perp^2)\pi + \lambda\pi(\sigma^2 + \pi^2)\} = 0, \quad (3.3)$$

$$\frac{1}{2L} \int_{-L}^L dx^- \{-(\mu^2 + \partial_\perp^2)\sigma + \lambda\sigma(\sigma^2 + \pi^2)\} = c. \quad (3.4)$$

These are easily obtained from the longitudinal integration of the Euler-Lagrange equation, due to the property $\int_{-L}^L dx^- \partial_- \partial_+ \phi = 0$. These relations are called the “zero-mode constraints”. Since they are due to the structure of the kinetic term, these constraints exist in any scalar models. When we have only scalar fields, we always have the zero-mode constraints.⁶ They are of special importance for the description of the symmetry breaking. To see this, let us decompose the longitudinal zero modes into c-number parts and (normal-ordered) operator parts,

$$\sigma_0 = \sigma_0^{(c)} + \sigma_0^{(op)}(\varphi_\sigma, \varphi_\pi), \quad (3.5)$$

$$\pi_0 = \pi_0^{(c)} + \pi_0^{(op)}(\varphi_\sigma, \varphi_\pi). \quad (3.6)$$

If the c-number parts of the solutions are nonvanishing, it directly follows that there are nonzero condensates: $\langle 0|\sigma|0\rangle = \langle 0|\sigma_0|0\rangle = \sigma_0^{(c)} \neq 0$ and $\langle 0|\pi|0\rangle = \langle 0|\pi_0|0\rangle = \pi_0^{(c)} \neq 0$. Therefore *it is necessary to find a nontrivial solution of the zero-mode constraint for describing the symmetry breaking*. Here, it is important to recall that only zero modes can give nonzero values to the scalar fields (cf. Sect. IID).

However, it is generally very difficult to solve the zero-mode constraint in quantum theory. The zero-mode constraints are nonlinear equations among operators, and thus we must face the notorious problem of operator ordering.

⁶Note that this does not necessarily hold if gauge fields couple to scalars, because the structure of the kinetic term changes. We discuss this in Sect. V.

Since the solution necessarily depends on the operator ordering, we must specify an appropriate ordering with some criterion. In many works involving the zero-mode constraints, people often choose, on general grounds, the Weyl ordering with respect to both constrained and unconstrained variables. However, it is not obvious whether the Weyl ordering in constraint equations makes sense. The dependent variables are written in terms of the independent variables, and thus if we substitute the solution into the constraint, the final ordering of the each term is no longer the Weyl ordering with respect to the independent variables. Instead, the most reliable criterion for determining the operator ordering is as follows. To make the discussion clear, let us consider a commutator $[\sigma_0, \varphi_\sigma]$. This can be evaluated in two different ways, (I) by using the solution $\sigma_0^{\text{sol}} = \sigma_0(\varphi_\sigma, \varphi_\pi)$ of the zero-mode constraint with the standard quantization rule ($i, j = \sigma, \pi$)

$$[\varphi_i(x), \varphi_j(y)]|_{x^+ = y^+} = -\frac{i}{4} \left\{ \epsilon(x^- - y^-) - \frac{x^- - y^-}{L} \right\} \delta_{ij} \delta^{(2)}(x_\perp - y_\perp), \quad (3.7)$$

or (II) by calculating the Dirac bracket $\{\sigma_0, \varphi_\sigma\}_D$ and transforming it into the quantum commutator. For case (I), we assume that the solution was obtained without using the commutator $[\sigma_0, \varphi_\sigma]$. When we solve the constraint, we must work with a specific operator ordering, and the result depends on the ordering we choose. For case (II), we must also determine the ordering on the right-hand side of the Dirac bracket $\{\sigma_0, \varphi_\sigma\}_D = \dots$. Thus, we have two ambiguities in the operator ordering: the ordering of the constraint equation in (I) and the ordering of the right-hand side of the Dirac bracket in (II). Since the methods (I) and (II) must give the same result for $[\sigma_0, \varphi_\sigma]$, the operator ordering should be imposed so that these two quantities are identical.⁷ In other words, we determine the operator ordering of the right-hand side in the Dirac brackets so that it coincides with the direct evaluation. This should be the criterion for an appropriate operator ordering. However, as may be expected, it is extremely difficult to find such a “consistent operator ordering”. Thus, practically, we just work with several particular orderings and compare the results to check the consistency.

In the sigma model with tree-level symmetry breaking, if we consider only the leading order of the semiclassical approximation, we do not have to worry about the operator ordering. Then the zero modes are just classical numbers to be determined by

$$\mu^2 \pi_0^{\text{cl}} + \lambda(\pi_0^{\text{cl}})^3 + \lambda \pi_0^{\text{cl}} (\sigma_0^{\text{cl}})^2 = 0, \quad (3.8)$$

$$\mu^2 \sigma_0^{\text{cl}} + \lambda(\sigma_0^{\text{cl}})^3 + \lambda \sigma_0^{\text{cl}} (\pi_0^{\text{cl}})^2 = c. \quad (3.9)$$

When $c = 0$, we have the usual symmetry breaking solution, $\sigma_0^{\text{cl}} = \sqrt{\mu^2/\lambda}$ and $\pi_0^{\text{cl}} = 0$. If we solve the zero-mode constraints perturbatively with respect to the coupling constant λ , we do not have a symmetry breaking solution. Because Eqs. (3.8) and (3.9) determine the value of the condensate, we can say that they are the gap equations. Indeed, the same equations are also obtained by differentiating the tree-level effective potential of spatially constant fields.

The lesson we learn from the above analysis is that to realize the symmetry breaking in the DLCQ approach, it is necessary to obtain a nontrivial solution to the zero-mode constraint. Since we have selected the potential so that it induces tree-level symmetry breaking, even the classical solution of the zero-mode constraint gives a nontrivial solution. However, in general, we have to solve the zero-mode constraint nonperturbatively at the quantum level.

B. The chiral Yukawa model – Scalar and fermionic fields

Now let us study a more complicated example, the chiral Yukawa model. We add an N -component fermion field Ψ_a ($a = 1, \dots, N$) that couples to the scalars via the Yukawa interaction:

$$\mathcal{L} = \bar{\Psi}_a (i\partial^\mu - m) \Psi_a + \frac{N}{2\mu^2} \{(\partial\sigma)^2 + (\partial\pi)^2\} - \frac{N}{2\lambda} (\sigma^2 + \pi^2) - (\sigma \bar{\Psi}_a \Psi_a + \pi \bar{\Psi}_a i\gamma_5 \Psi_a). \quad (3.10)$$

Here μ is a dimensionless parameter and the fermionic field is assumed to be antiperiodic in the x^- direction. This model is not a simple extension of the previous sigma model. Note that the Lagrangian (3.10) does not have a quartic

⁷If we cannot solve the constraint without knowing the commutator $[\sigma_0, \varphi_\sigma]$, then we must determine the operator ordering *self-consistently*.

interaction of the scalars. Therefore, this model undergoes chiral symmetry breaking at the quantum level. This can be explicitly verified in a conventional calculation with the effective potential. Also this may be seen clearly by the fact that the model becomes the NJL model in the infinitely heavy mass limit for scalars $\mu \rightarrow \infty$ and that the NJL model exhibits $D\chi SB$ at the quantum level.⁸ The NJL model is discussed in the next subsection.

As we have scalar fields, let us use the DLCQ approach again. There is one additional complication here that does not exist in the previous case. In addition to the two zero-mode constraints for σ_0 and π_0 ,

$$\left(\frac{\mu^2}{\lambda} - \partial_\perp^2\right) \begin{pmatrix} \sigma_0 \\ \pi_0 \end{pmatrix} + \frac{\mu^2}{N} \frac{1}{\sqrt{2}} \int_{-L}^L \frac{dx^-}{2L} \left[\psi_a^\dagger \begin{pmatrix} 1 \\ -i\sigma_3 \end{pmatrix} \chi_a + \chi_a^\dagger \begin{pmatrix} 1 \\ i\sigma_3 \end{pmatrix} \psi_a \right] = 0, \quad (3.11)$$

we have one fermionic constraint for the bad component χ ,

$$i\partial_- \chi_a(x) = \frac{1}{\sqrt{2}} (-\sigma^i \partial_i + m + \sigma(x) + i\pi(x)\sigma_3) \psi_a. \quad (3.12)$$

These three equations form a coupled set for the dependent variables σ_0 , π_0 and χ to be represented by the other independent variables, φ_σ , φ_π and ψ .

It is not difficult to obtain *classical* solutions to the above equations. To solve them classically, we treat the fermionic fields just as Grassmann numbers. The classical solutions enable us to check explicitly whether the system with a *mass term* is invariant or not under the LF chiral transformation. It turns out that chiral symmetry exists if and only if $m = 0$ and that the scalars and fermions transform as in Eqs. (2.18) and (2.8). Therefore, we do not have to consider the extra symmetry which is present in the massive free fermion theory. Now that we can treat the LF chiral transformation as being equivalent to the usual one, the chiral current is also given as usual. Note that the LF chiral charge defined by spatial integration of its plus component is written only in terms of independent degrees of freedom (here, φ_σ , φ_π , and ψ), which should be the case, by definition. We do not discuss about the classical analysis any further here. Explicit demonstrations involving the classical solutions and their consequences are found in Ref. [13]. Below, we discuss similar situations in more detail in the NJL model.

Now let us turn to the quantum theory. Unlike the classical theory, it is extraordinarily difficult to solve the constraints in this case as operator equations. The primary reason for this is the problem of operator ordering. Probably the best way to attack this problem is to find a consistent operator ordering, as we discussed in the case of sigma model. This is, however, a very difficult task in our model. We, rather, consider only a specific ordering. It can be shown that the leading order (in the $1/N$ expansion) result does not change even if we choose other operator orderings.

We must solve the coupled equations (3.11) and (3.12) of the dependent variables χ , σ_0 and π_0 . First of all, it is easy to remove χ from them. Solving the fermionic constraint (3.12) formally by inverting $i\partial_-$ and then inserting the solution χ into Eq. (3.11), we find complicated equations for the scalar zero modes σ_0 and π_0 . After changing the ordering, so that the leading order calculation becomes relatively easy, we obtain

$$\begin{aligned} & \left(\frac{\mu^2}{\lambda} - \partial_\perp^2\right) \begin{pmatrix} \sigma_0 \\ \pi_0 \end{pmatrix} \\ & + \frac{\mu^2}{4N} \int_{-L}^L \frac{dx^- dy^-}{2L} \frac{\epsilon(x^- - y^-)}{2i} \left\{ \psi_x^{a\dagger} \begin{pmatrix} -1 \\ i\sigma_3 \end{pmatrix} \sigma^i \partial_i \psi_y^a - \partial_i \psi_y^{a\dagger} \begin{pmatrix} -1 \\ i\sigma_3 \end{pmatrix} \sigma^i \psi_x^a \right. \\ & \quad \left. + \begin{pmatrix} m + \sigma(y) \\ \pi(y) \end{pmatrix} (\psi_x^{a\dagger} \psi_y^a - \psi_y^{a\dagger} \psi_x^a) - \begin{pmatrix} -\pi(y) \\ m + \sigma(y) \end{pmatrix} (\psi_x^{a\dagger} i\sigma_3 \psi_y^a + \psi_y^{a\dagger} i\sigma_3 \psi_x^a) \right\} \\ & + \text{H.c.} = 0, \end{aligned} \quad (3.13)$$

where we use the definitions $\psi_y^a = \psi^a(y^-, x_\perp)$, $\sigma(y) = \sigma_0(x_\perp) + \varphi_\sigma(y^-, x_\perp)$, and so on. (Do not confuse the Pauli matrices σ^i with the scalar field $\sigma(x)$.)

We now show how to determine the leading order solution of the zero-mode constraints in the $1/N$ expansion: $\sigma_0 = \sigma_0^{(c)} + \sigma_0^{(\text{op})} = \mathcal{O}(N^0)$ and $\pi_0 = \pi_0^{(c)} + \pi_0^{(\text{op})} = \mathcal{O}(N^0)$. First of all, the c-number part of the zero modes can be chosen as $\sigma_0^{(c)} \neq 0$ and $\pi_0^{(c)} = 0$, because we confirmed in the classical analysis that (σ_0, π_0) rotates chirally in the massless case. Because of the relation which holds in leading order, $\langle \sigma \rangle = -\frac{\Delta}{N} \langle \bar{\Psi} \Psi \rangle$ and $\langle \pi \rangle = -\frac{\Delta}{N} \langle \bar{\Psi} i\gamma_5 \Psi \rangle$, a nonzero

⁸Furthermore, the chiral Yukawa model in the mean field approximation is equivalent to the NJL model.

value of $\sigma_0^{(c)}$ immediately implies a nonzero fermion condensate. Taking the VEV of the zero-mode constraint for σ , we find

$$M - m = 2\lambda M \int \frac{d^2 p_\perp}{(2\pi)^2} \sum_{n=\frac{1}{2}, \frac{3}{2}, \dots} \frac{1}{p_n^+} \frac{\Delta p^+}{2\pi}, \quad (3.14)$$

where $\Delta p^+ = \pi/L$ and $p_n^+ = \pi n/L$, and we have introduced $M \equiv m + \sigma_0^{(c)}$. Physically, this equation should be the gap equation, by which we can determine the condensate $\sigma_0^{(c)}$, and equivalently the physical fermion mass M . However, it is not evident whether we can regard it as the gap equation, because Eq. (3.14) in the naive chiral limit $m \rightarrow 0$ cannot give nonzero M . The flaw of this observation lies in the loss of mass dependence in the infinite summation of Eq. (3.14). It turns out [10] that this can be fixed by supplying mass information properly through a cutoff when we regularize the divergent summation (see Sect. IVA for more details). For example, if we adopt a cutoff which respects the parity invariance,⁹ the infinite momentum sum can be approximated by an integral over p^+ with the integration range $(M^2 + p_\perp^2)/2\Lambda < p^+ < \Lambda$. Consequently, Eq. (3.14) becomes *dependent* on the mass M and can be considered as a gap equation:

$$M - m = \lambda M \frac{\Lambda^2}{4\pi^2} \left\{ 2 - \frac{M^2}{\Lambda^2} \left(1 + \ln \frac{2\Lambda^2}{M^2} \right) \right\}. \quad (3.15)$$

Indeed, even in the chiral limit $m \rightarrow 0$, this equation is a nonlinear equation in M , and when the coupling constant λ is larger than the critical value $\lambda_{\text{cr}} = 2\pi^2/\Lambda^2$, we have a nontrivial solution $M = M_0 \neq 0$. This also implies that the c-number part of σ_0 has been determined to be $\sigma_0^{(c)} = M - m$.

When we derived the above cutoff, we used the dispersion relation with the dynamical fermion mass M . That is, we used $2p^+p^- - \mathbf{p}_\perp^2 = M^2$ instead of $2p^+p^- - \mathbf{p}_\perp^2 = m^2$. This is an important step in obtaining the gap equation, and it corresponds to imposing the self-consistency. As we see below, the use of the physical fermion mass is natural, since the prefactor of the integral in Eq. (3.14) is M .

The operator parts of the zero modes are obtained by “linearization” of the zero-mode equations to leading order in $1/N$ expansion. The result is

$$\begin{pmatrix} \sigma_0^{(\text{op})} \\ \pi_0^{(\text{op})} \end{pmatrix} = -\frac{\mu^2}{N} (m_{\text{ZM}}^2 - \partial_\perp^2)^{-1} \frac{1}{2L} \int_{-L}^L dx^- \left[: \bar{\Psi}_M^a \begin{pmatrix} 1 \\ i\gamma_5 \end{pmatrix} \Psi_M^a : \right], \quad (3.16)$$

where $m_{\text{ZM}}^2 = \mu^2 m / (\lambda M)$ and Ψ_M is a free fermion operator with mass M :

$$\Psi_M = 2^{-\frac{1}{4}} \begin{pmatrix} \psi \\ \chi_M \end{pmatrix} = 2^{-\frac{1}{4}} \begin{pmatrix} \psi \\ \frac{1}{\sqrt{2}i\partial_-} (-\sigma^i \partial_i + M) \psi \end{pmatrix}. \quad (3.17)$$

Finally, by inserting the c-number and operator parts of σ_0 and π_0 into the solution for the fermionic constraint, we obtain the bad component of the fermion:

$$\chi = \chi_M + \frac{1}{\sqrt{2}} \frac{1}{i\partial_-} \left(\sigma_0^{(\text{op})} - \pi_0^{(\text{op})} i\sigma_3 \right) \psi. \quad (3.18)$$

Equations (3.16) and (3.18) and the c-number part $\sigma_0^{(c)} \neq 0$, $\pi_0^{(c)} = 0$ are the leading-order solutions to the coupled equations. If we select the nontrivial solution (i.e., a solution $M \neq 0$ even in the chiral limit) of the gap equation, then the resulting theory should describe the broken phase. The trivial solution gives the symmetric phase. Therefore, the resulting theories written in terms of independent variables change according to the solutions we choose. Various consequences of this unusual situation are discussed in the next section.

In summary, what we found in this analysis is that (A) the zero-mode constraints are very important, since they contain information about the gap equation, and that (B) we have to be careful about the divergent summation over the longitudinal momenta to get a nontrivial gap equation. The second point is discussed in more detail in the next section. The first point, that is, the metamorphosis of the zero-mode constraint into the gap equation, has already been seen in the sigma model. Here we find a situation similar to that in the more complicated system with chiral symmetry. The appearance of the gap equation is more nontrivial in this model in the sense that we need a quantum and nonperturbative analysis and a prescription for the mass information loss.

⁹The parity transformation is given by $p^+ \leftrightarrow p^-$, $p_\perp \rightarrow -p_\perp$.

C. The Nambu–Jona-Lasinio model – Fermionic fields

The last example in this section is the NJL model, [23] which is a fermion theory with four-Fermi interaction:

$$\mathcal{L} = \bar{\Psi}(i\cancel{d} - m)\Psi + \frac{\lambda}{2N} [(\bar{\Psi}\Psi)^2 + (\bar{\Psi}i\gamma_5\Psi)^2] . \quad (3.19)$$

This model is the most well-known classic example exhibiting chiral symmetry breaking. [24] The chiral symmetry that is present in the massless Lagrangian dynamically breaks down due to a nonzero condensate $\langle\bar{\Psi}\Psi\rangle \neq 0$. Recently, there have been several efforts to reproduce this in the LF NJL model. [25,11,12,14,26] Here we explain one of such attempts based on our studies. [11,12,14]

Since we do not have scalar fields, the method based on the zero-mode constraint is useless in this model. However, it is very suggestive to consider the chiral Yukawa model from a different point of view. Recall that the first task faced in the application of that model was to remove the bad component from the coupled constraint equations Eqs. (3.11) and (3.12). This led us to the zero-mode constraints (3.13), and we successfully extracted the gap equation from them. However, we should note that we could have removed the zero modes first (instead of the bad spinor component) from the coupled equations. Indeed we can formally solve the zero-mode constraints (3.11) and substitute the solutions into the fermionic constraint (3.12). Then *the resulting fermionic constraint must carry information concerning the gap equation*, because the information should be preserved. What does this tell us about the NJL model? The LF NJL model has one complicated fermionic constraint that is immediately obtained as the Euler-Lagrange equation for χ ,

$$i\partial_- \chi_a = \frac{1}{\sqrt{2}} (-\sigma^i \partial_i + m) \psi_a - \frac{\lambda}{2N} \left\{ \psi_a \left(\psi_b^\dagger \chi_b + \chi_b^\dagger \psi_b \right) + \sigma^3 \psi_a \left(\psi_b^\dagger \sigma^3 \chi_b - \chi_b^\dagger \sigma^3 \psi_b \right) \right\} , \quad (3.20)$$

where summation over the “color” and spinor indices is implied. The above discussion for the chiral Yukawa model suggests that this fermionic constraint contains the information about the gap equation. We will see that this is indeed the case. In particular, this can be easily confirmed if we note the equivalence of the NJL model and the heavy mass limit of the chiral Yukawa model. Here, we do not use the DLCQ method, as we have no scalar fields. The longitudinal extension is not compactified, but we impose the antiperiodic boundary condition in this direction $\Psi(x^- = -\infty) = -\Psi(x^- = \infty)$.

1. Classical solution of the fermionic constraint

Before proceeding to quantum analysis, let us consider the classical solution of the fermionic constraint. Since we can treat all the variables as Grassmann numbers in the classical analysis, the equation becomes tractable, and it is not difficult to solve it. Indeed, the exact solution with antiperiodic boundary condition is given by

$$\begin{pmatrix} \chi_{1a}(x) \\ -\chi_{2a}^\dagger(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} dy^- G_{ab}(x^-, y^-, x_\perp) \begin{pmatrix} m\psi_{1b}(y^-) - \partial_z \psi_{2b}(y^-) \\ -\partial_z \psi_{1b}^\dagger(y^-) + m\psi_{2b}^\dagger(y^-) \end{pmatrix} , \quad (3.21)$$

where $\partial_z = \partial_1 - i\partial_2$, and the “Green function” $G_{ab}(x^-, y^-, x_\perp)$ is

$$\begin{aligned} G_{ab}(x^-, y^-, x_\perp) &= G^{(0)}(x) \left[\frac{1}{2i} \epsilon(x^- - y^-) + C \right] G^{(0)}(y)^{-1} , \\ G^{(0)}(x) &= P e^{i \frac{\lambda}{N} \int_{-\infty}^{x^-} \mathcal{A}(y^-) dy^-} , \quad \mathcal{A}_{ijab} = \begin{pmatrix} \psi_{1a} \psi_{1b}^\dagger & \psi_{1a} \psi_{2b} \\ \psi_{2a}^\dagger \psi_{1b}^\dagger & \psi_{2a}^\dagger \psi_{2b} \end{pmatrix} . \end{aligned}$$

The integral constant C is determined so that the solution satisfies antiperiodic boundary condition. The symbol “P” in the definition of $G^{(0)}(x)$ represents the path-ordered product.

The most significant benefit of having the exact form of the classical solution is that we can explicitly check the transformation of the bad component under the LF chiral transformation (2.6). Recall the importance of the classical solution in the massive free fermion theory in Sect. IIB. We apply similar analysis here. Decomposition of χ is straightforward:

$$\begin{pmatrix} \chi_{1a}^{(0)} \\ -\chi_{2a}^{(0)\dagger} \end{pmatrix} = -\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} dy^- G_{ab}(x^-, y^-, x_\perp) \begin{pmatrix} \partial_z \psi_{2b} \\ \partial_z \psi_{1b}^\dagger \end{pmatrix}, \quad (3.22)$$

$$\begin{pmatrix} \chi_{1a}^{(m)} \\ -\chi_{2a}^{(m)\dagger} \end{pmatrix} = \frac{m}{\sqrt{2}} \int_{-\infty}^{\infty} dy^- G_{ab}(x^-, y^-, x_\perp) \begin{pmatrix} \psi_{1b} \\ \psi_{2b}^\dagger \end{pmatrix}. \quad (3.23)$$

Since the matrix \mathcal{A} , and thus $G_{ab}(x, y)$, is invariant under the transformation (2.6), it is easy to find that $\chi^{(0)}$ and $\chi^{(m)}$ transform as in Eq. (2.7). Therefore, if $m = 0$, the LF chiral transformation (2.6) is equivalent to the usual chiral transformation. The chiral current and the chiral charge are given by Eqs. (2.11) and (2.12), respectively.

Hence the massless case is seen to be quite satisfactory. Then, what about the massive case? As we discussed in Sect. IIB, the chiral symmetry survives even with a mass term in the free fermion case. We must bear in mind such a possibility even in the NJL model. Thus it is worthwhile to check whether the *massive* NJL model is invariant under the LF chiral transformation. To see this, it is convenient to treat the Hermite Lagrangian

$$\mathcal{L}_{\text{Hermite}} = \frac{1}{2} i \psi^\dagger \overleftrightarrow{\partial}_+ \psi - \frac{1}{2\sqrt{2}} [(\psi^\dagger \sigma^i \partial_i \chi + \partial_i \chi^\dagger \sigma^i \psi) + m (\psi^\dagger \chi + \chi^\dagger \psi)].$$

Note that this is equivalent to the free Lagrangian, except that χ is a solution of Eq. (3.20). Now, the apparently non-invariant term depends linearly on m :

$$-\frac{1}{2\sqrt{2}} \psi^\dagger (\sigma^i \partial_i \chi^{(m)} + m \chi^{(0)}) + \text{H.c.}.$$

In the massive free fermion case, we had the same term, but it eventually vanished due to Eq. (2.9). However, in the NJL model, it is evident from Eqs. (3.22) and (3.23) that such a relation does not hold, because G depends on x_\perp . Therefore we have verified that the massive NJL model is *not* invariant under the LF chiral transformation. If and only if $m = 0$, the LF chiral transformation is the symmetry of the NJL model and is equivalent to the usual chiral transformation. This is, of course, not a surprising result, but it must be checked explicitly.

Irrespective of whether we have a mass term or not, we always use the definition for the chiral current Eq. (2.11). In the massless case, it is, of course, a conserved current $\partial_\mu J_5^\mu = 0$, while in the massive case, the usual relation $\partial_\mu J_5^\mu = 2m\Psi i\gamma_5\Psi$ holds, which is derived by using the Euler-Lagrange equation of the massive NJL model.

2. Quantum and nonperturbative solution

Now let us turn to the quantum aspects of the model. We solve the nonlinear operator equation (3.20) by using the $1/N$ expansion. This entails two additional problems, as well as the usual operator ordering problem. The first problem is determining how to count the order $\mathcal{O}(N^n)$ of an operator instead of its matrix element. The second is that it is physically difficult to justify the $1/N$ expansion of the fermionic field itself. To overcome these, in Ref. [14], we rewrote the fermionic constraint in terms of bilocal fields and solved them with fixed operator ordering. More precisely, we introduced three bilocal fields defined by

$$\begin{aligned} \mathcal{M}_{\alpha\beta}(\mathbf{x}, \mathbf{y}) &= \sum_{a=1}^N \psi_\alpha^{a\dagger}(x^+, \mathbf{x}) \psi_\beta^a(x^+, \mathbf{y}), \\ \mathcal{T}_{\alpha\beta}(\mathbf{x}, \mathbf{y}) &= \frac{1}{\sqrt{2}} \sum_{a=1}^N \left(\psi_\alpha^{a\dagger}(x^+, \mathbf{x}) \chi_\beta^a(x^+, \mathbf{y}) + \chi_\beta^{a\dagger}(x^+, \mathbf{y}) \psi_\alpha^a(x^+, \mathbf{x}) \right), \\ \mathcal{U}_{\alpha\beta}(\mathbf{x}, \mathbf{y}) &= \frac{-i}{\sqrt{2}} \sum_{a=1}^N \left(\psi_\alpha^{a\dagger}(x^+, \mathbf{x}) \chi_\beta^a(x^+, \mathbf{y}) - \chi_\beta^{a\dagger}(x^+, \mathbf{y}) \psi_\alpha^a(x^+, \mathbf{x}) \right). \end{aligned}$$

The $1/N$ expansion of the bilocal operator $\mathcal{M}_{\alpha\beta}(\mathbf{x}, \mathbf{y})$ is known as the Holstein-Primakoff expansion, which is a special case of the boson expansion method. [27] Finding $\mathcal{T}_{\alpha\beta}$ and $\mathcal{U}_{\alpha\beta}$ corresponds to obtaining χ and χ^\dagger . At each order of the $1/N$ expansion, the bilocal fermionic constraints are linear with respect to $\mathcal{T}_{\alpha\beta}$ and $\mathcal{U}_{\alpha\beta}$, and thus are easily solved. [14]

In particular, the leading order of the bilocal fermionic constraint reduces to the very simple form

$$M - m = \lambda M \int \frac{d^3 p}{(2\pi)^3} \frac{\epsilon(p^+)}{p^+}, \quad (3.24)$$

where we have defined the physical fermion mass $M = m - (\lambda/N)\langle\bar{\Psi}\Psi\rangle$, and $\langle\bar{\Psi}\Psi\rangle$ is the leading order of $\mathcal{T}_{\alpha\alpha}(\mathbf{x}, \mathbf{x}) = \mathcal{O}(N)$. This is essentially the same as Eq. (3.14), and therefore it becomes the gap equation (3.15) if we use the same cutoff scheme. The solution of Eq. (3.24) gives the lowest order of $\mathcal{T}_{\alpha\alpha}(\mathbf{x}, \mathbf{x})$. Similarly, higher order contributions are determined order by order. Once we find solutions of the fermionic constraints, the Hamiltonian, which is also written in terms of the bilocal fields, is obtained. If we choose a nontrivial solution of the gap equation, the resulting Hamiltonian describes the broken phase even with a trivial vacuum.

In this last example, we have explained that the physical role of the fermionic constraint is very similar to that of the zero-mode constraint in the previous examples. We have seen a close parallel between these two constraints. In particular, it should be noted that the gap equation results from the longitudinal zero mode of the bilocal fermionic constraint. It is very natural that we can realize the broken phase by solving the quantum fermionic constraint with the $1/N$ expansion, because the fermionic constraint is originally a part of the Euler-Lagrange equation and thus must include relevant information regarding the dynamics. What we have done here is actually very similar to the usual mean-field approximation for the Euler-Lagrange equations. Indeed, the leading order in the $1/N$ expansion corresponds to the mean-field approximation. However, our method of solving the fermionic constraint with the boson expansion method can easily go beyond the mean-field level. Such a higher-order calculation enables us to derive the correct broken Hamiltonian, and so on. These points are discussed in Ref. [14] in more detail.

IV. CHIRAL SYMMETRY BREAKING ON THE LIGHT FRONT

In this section we discuss some unusual but important aspects of the methods of realizing chiral symmetry breaking on the LF. We found in the previous section that the chiral condensate can be obtained from the characteristic constraint equations. In spite of having a nonzero chiral condensate, our vacuum is still the trivial Fock vacuum. This coexistence of the chiral condensate and the trivial vacuum leads to various unusual consequences. After we make a more detailed investigation of the chiral condensate, and, in particular, the problem of the loss of mass information, we introduce the concept of multiple Hamiltonians in the broken phase and show how the Nambu-Goldstone bosons appear. Then, we discuss the most important properties of the broken phase, that is, the peculiarity of the chiral transformation and the nonconservation of the chiral charge.

A. Chiral condensate and problem of mass information loss

Let us first consider the reasons why we succeeded in obtaining a nonzero fermion condensate. To this end, it is very instructive to see the fermionic constraint (3.20) from a different point of view. Rewriting the fermionic constraint as

$$i\partial_{-}\chi_a = \frac{1}{\sqrt{2}} (-\sigma^i \partial_i + m) \psi_a - \frac{\lambda}{\sqrt{2}N} (\psi_a \mathcal{T}_{\alpha\alpha}(\mathbf{x}, \mathbf{x}) + i\sigma^3 \psi_a \sigma_{\alpha\beta}^3 \mathcal{U}_{\alpha\beta}(\mathbf{x}, \mathbf{x})) ,$$

and substituting the leading order solutions into the bilocal operators $\mathcal{T}_{\alpha\alpha}(\mathbf{x}, \mathbf{x})$ and $\mathcal{U}_{\alpha\beta}(\mathbf{x}, \mathbf{x})$, we find that the leading order equation turns out to be equivalent to the fermionic constraint for a free fermion with mass M :

$$i\partial_{-}\chi_a = \frac{1}{\sqrt{2}} (-\sigma^i \partial_i + M) \psi_a . \quad (4.1)$$

Also at the same order, the equation of motion for the good component ψ tells us that the fermion acquires a mass M . This procedure is actually equivalent to the mean-field approximation in the Euler-Lagrange equation. The same gap equation as before can be derived as a self-consistency equation $M = m + (\lambda/N)\langle\bar{\Psi}\Psi\rangle$: [25]

$$\begin{aligned} M &= m + \frac{\lambda}{N} \cdot \frac{1}{\sqrt{2}} \langle 0 | (\psi^\dagger \chi + \chi^\dagger \psi) | 0 \rangle \\ &= m + \frac{\lambda}{N} \cdot \frac{M}{2} \langle 0 | \left\{ \psi^\dagger \frac{1}{i\partial_{-}} \psi - \left(\frac{1}{i\partial_{-}} \psi^\dagger \right) \psi \right\} | 0 \rangle \\ &= m - \frac{\lambda}{N} \cdot \frac{NM}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{\epsilon(p^+)}{p^+}. \end{aligned} \quad (4.2)$$

In the last line, we have followed the standard canonical LF quantization with the mode expansion (2.14). Two things are important here. First, the right-hand side of the gap equations (3.14) or (3.24) comes essentially from the vacuum expectation value of $\bar{\Psi}\Psi$ in a free massive theory. Second, the fermion condensate $\langle\bar{\Psi}\Psi\rangle$ is not zero for a massive fermion even if the vacuum is trivial.

Recall that Eq. (4.2) cannot be the gap equation as it stands, because it does not have the correct mass dependence. Therefore, we can now understand the essential question to be asked: How can we compute $\langle 0 | \bar{\Psi}\Psi | 0 \rangle$ correctly in the LF formalism? This is not a problem limited to chiral symmetry breaking, but is more general. Indeed, it is a common problem for the computation of more than two-point functions. The case we encounter here is a special case of the fermion bilinears. Let us consider this situation more rigorously. In the massive free fermion theory, we can directly compute the two-point functions of the fermion bilinears. For example, the usual ET quantization scheme gives [28]

$$\begin{aligned} \langle 0 | \bar{\Psi}(x)\Psi(0) | 0 \rangle &= \int_0^\infty d\kappa^2 w_{(1)}(\kappa^2) \Delta^{(+)}(x, \kappa^2) \\ &= M \Delta^{(+)}(x, M^2) \\ &= \frac{M^2}{4\pi^2} \frac{K_1(M\sqrt{-x^2})}{\sqrt{-x^2}} \end{aligned} \quad (4.3)$$

for $x^2 < 0$. Here we have used the expression for the free fermion spectrum $w_{(1)}(\kappa^2) = M\delta(\kappa^2 - M^2)$, and $\Delta^{(+)}(x, \kappa^2)$ is one of the invariant delta functions. Restriction of Eq. (4.3) to the case of equal LF time, i.e. $x^+ \rightarrow 0$ yields ($x^2 = -x_\perp^2 < 0$)

$$\langle 0 | \bar{\Psi}(x)\Psi(0) | 0 \rangle |_{x^+=0} = \frac{M^2}{4\pi^2} \frac{K_1(M\sqrt{x_\perp^2})}{\sqrt{x_\perp^2}}. \quad (4.4)$$

The chiral condensate is obtained as the limit $x_\perp \rightarrow 0$. It is evident that the result depends on the mass M nontrivially.

However, the LF canonical procedure gives only a trivial dependence of M , as is seen in Eq. (4.2). The origin of this discrepancy can be traced back to the integral representation of $\Delta^{(+)}(x, M^2)$. With the LC coordinates, it is given by

$$\Delta^{(+)}(x, M^2) = \frac{1}{(2\pi)^3} \int_0^\infty \frac{dp^+}{p^+} \int_{-\infty}^\infty d^2 p_\perp e^{-i\left(\frac{p_\perp^2 + M^2}{2p^+} x^+ + p^+ x^- - p_\perp^i x_\perp^i\right)}. \quad (4.5)$$

Clearly, $p^+ = 0$ is a singular point, and taking x^+ to be nonzero safely regulates this singularity. Therefore, a nontrivial M dependence remains after the p^+ integration, as is seen in Eq. (4.4). On the other hand, if we first set $x^+ = 0$ in the integral, the result becomes (erroneously) independent of M . This corresponds to the result of the naive LF quantization, which is formulated on the surface $x^+ = 0$. We thus see that the p^+ integration does not commute with the LC restriction $x^+ \rightarrow 0$. We must be careful when we encounter singularities of the type $1/p^+$, and have to resort to some prescription for treating them in the canonical LF quantization scheme. This problem was first recognized by Nakanishi and Yamawaki [29] long ago in the context of the scalar theories [because $\langle 0 | \phi(x)\phi(0) | 0 \rangle = \Delta^{(+)}(x, M^2)$] and recently by ourselves [10–14] in fermionic theories (see also Ref. [30]).

To remedy this problem within the framework of the canonical LF quantization, it is quite natural to supply the mass dependence as the regularization of the divergent integral $\int dp^+/p^+$:

$$\begin{aligned} \langle 0 | \bar{\Psi}(0)\Psi(0) | 0 \rangle &= -\frac{M}{(2\pi)^3} \int_0^\infty \frac{dp^+}{p^+} \int_{-\infty}^\infty d^2 p_\perp \\ &\rightarrow -\frac{M}{(2\pi)^3} \int_0^\infty \frac{dp^+}{p^+} \int_{-\infty}^\infty d^2 p_\perp f_\Lambda(p, M), \end{aligned} \quad (4.6)$$

where $f_\Lambda(p, M)$ is a cutoff function for the IR singularity. Since the prefactor of the integral tells us the mass of the system, we use M in the cutoff function. From our point of view, we do not know how to determine the appropriate form of the cutoff function $f_\Lambda(p, M)$. In fact, there are many possibilities that give the same result if renormalization is done properly. For example, Eq. (4.5) suggests the heat kernel regularization [10] (as seen by replacing x^+ by the imaginary time $-i\tau$). Other schemes are the three- or four-momentum cutoff [25] $|p| < \Lambda_3$, $p^2 < (\Lambda_4)^2$, and the parity invariant cutoff [10–14] $p^\pm < \Lambda_{\text{PI}}$, which we used in Sect. III. When we rewrite these conditions in terms of the LC variables p^+ and p_\perp , we necessarily use the dispersion relation. Thus the mass dependence enters into the cutoff functions $f_\Lambda(p, M)$, as we saw in Sect. IIIB in relation to Eq. (3.15). The parity invariant cutoff is an economical and

tractable scheme. All of these cutoff schemes are related to the symmetries which the system should have. Cutoff schemes without symmetry considerations could give erroneous results even if they contain the mass dependence. [31]

One thing to be noted is the peculiarity of the mode expansion (2.14). It is evident that the mode expansion contains no mass dependence. Essentially, it is this (mass-independent) mode expansion that caused the above encountered difficulties in the chiral condensate. On the other hand, such independence of the mass, in turn, implies that our mode expansion allows fermions with *any value of the mass*. In other words, the LF vacuum does not distinguish the value of the fermion mass. Therefore we can regard the vacuum of a massless fermion as that of a massive one. The mass of the fields is determined by the Hamiltonian. This is the reason that we can have the trivial vacuum while having a nonzero fermion condensate. This phenomenon is not limited to our specific mode expansion but is common to all LF field theories. Indeed, even if we expand a fermion field with free spinor wave functions $u(p)$ and $v(p)$, we have no mass dependence. [32]

B. Multiple Hamiltonians

In the previous section, we saw that the gap equations are obtained from the constraint equations characteristic of the LF formalism. In general, a gap equation for an order parameter allows several independent solutions corresponding to different phases. This also implies that there are several solutions to one constraint equation. Selecting a nontrivial value of the order parameter corresponds to selecting a specific solution of the constraint. If we substitute the solution into the Hamiltonian, the Hamiltonian depends on the solutions we choose. In other words, we have multiple Hamiltonians corresponding to possible solutions of the constraint. This was shown explicitly both in scalar models [33] and in a fermionic model. [14]

Let us explain this point more concretely. Suppose we have constraints characteristic of the LF formalism

$$\Omega_i^{\text{LF}}[\xi_i, \eta_i] = 0,$$

where the $\xi_i(\eta_i)$ represent independent (dependent) variables. In the previous examples, the constrained variables η_i were the bad component of a fermion and the zero modes of the scalar fields. By solving these equations, we obtain two kinds of solutions, symmetric ones $\eta_i = \eta_i^{\text{sym}}[\xi_i]$ and broken ones $\eta_i = \eta_i^{\text{br}}[\xi_i]$. Since the original canonical Hamiltonian is a function of both ξ_i and η_i , we have different Hamiltonians corresponding to the different solutions:

$$\begin{aligned} H_{\text{sym}}[\xi_i] &= H[\xi_i, \eta_i = \eta_i^{\text{sym}}], \\ H_{\text{br}}[\xi_i] &= H[\xi_i, \eta_i = \eta_i^{\text{br}}]. \end{aligned}$$

This contrasts sharply with the usual description of symmetry breaking. In the standard description, we have multiple vacua but only one Hamiltonian. However, in our case, the vacuum remains the trivial Fock vacuum, but we have multiple Hamiltonians. Therefore *the information concerning the nontrivial vacuum structure in the usual scheme is moved into the Hamiltonian in the LF formalism*. This is a very unique way of realizing chiral symmetry breaking, and it inevitably causes various unusual situations, as we discuss below.

Once we have the Hamiltonian, we can compute the wavefunctions and mass spectra of excited states by solving the LF bound state equation. This is one of the important aspects of the LF Hamiltonian formalism. For example, mesonic states in the broken phase can be obtained by using a Hamiltonian in the broken phase H_{br} as

$$H_{\text{br}}|\text{meson}\rangle = \frac{P_\perp^2 + M_{\text{meson}}^2}{2P^+}|\text{meson}\rangle, \quad (4.7)$$

where $|\text{meson}\rangle$ can be expanded in terms of various Fock states:

$$|\text{meson}\rangle = |q\bar{q}\rangle + |q\bar{q}g\rangle + |q\bar{q}q\bar{q}\rangle + \dots$$

As we discussed in the Introduction, several kinds of nonperturbative techniques are useful for solving the above equation. In particular, due to the simplicity of the vacuum, it is natural to expect that lower excited states can be economically described by a few number of the Fock states with only a few constituents. For example, the Tamm-Danoff approximation, which truncates the Fock state expansion to a few terms, is effective in the LF quantization.

One important problem is how to determine the true Hamiltonian from several possible Hamiltonians. Since we do not compute the effective potential, it seems that we do not have any criterion to select the correct Hamiltonian. However, if we use the wrong Hamiltonian in the LF bound state equation, its solutions will include a tachyonic state with negative mass square. This can be a rule for eliminating wrong Hamiltonians. We comment on this in the next subsection in the context of a concrete example.

C. The NG bosons

Now let us turn other important physical phenomena in the broken phase, those of the NG bosons. The NG theorem tells us that if a continuous symmetry breaks down spontaneously, there always appears a massless state associated with the broken symmetry. The massless nature of the NG boson is ensured as long as the symmetry is exact at the Lagrangian level. However, this is a bad news for the LF formalism. It is known that it fails to describe a part of the dynamics of massless particles because they can propagate parallel to the $x^+ = \text{const}$ surface. To avoid this, in the previous three examples of Sect. III, we always worked with explicit symmetry breaking terms: $c \neq 0$ in the sigma model and a nonzero bare mass term in the others. A small explicit symmetry breaking term gives a small nonzero mass to the NG boson and thus cures the situation. In the following, we show how to obtain the (pseudo) NG boson (with nonzero mass) in the NJL model. The problems arising from the massless nature of the NG boson are commented on below.

As discussed in Sect. IVB, a mesonic state can be approximated well by a few constituents. This is true of a pseudoscalar pionic state in the NJL model,¹⁰ which should be the NG boson associated with the breaking of chiral symmetry. Let us consider a pion as the lowest Fock component $|q\bar{q}\rangle$ with one fermion (b^\dagger) and one antifermion (d^\dagger) created on the Fock vacuum

$$|\pi; \mathbf{P}\rangle = \frac{1}{\sqrt{N}} \int_0^{P^+} dk^+ \int_{-\infty}^{\infty} d^2 k_\perp \Phi^{\alpha\beta}(\mathbf{k}) b_\alpha^{a\dagger}(\mathbf{k}) d_\beta^{a\dagger}(\mathbf{P} - \mathbf{k}) |0\rangle, \quad (4.8)$$

where we have used the mode expansion (2.14), and $\Phi^{\alpha\beta}(\mathbf{k})$ is the LC wavefunction normalized as

$$\int_0^1 dx \int \frac{d^2 k_\perp}{16\pi^3} \sum_{\alpha\beta} |\Phi^{\alpha\beta}(\mathbf{k})|^2 = 1. \quad (4.9)$$

We can determine $\Phi^{\alpha\beta}(\mathbf{k})$ and the pion mass m_π by solving the LF bound state equation (4.7). Since we are interested in the broken phase, we assume that the coupling is larger than the critical coupling $\lambda > \lambda_{\text{cr}}$, and we select the broken phase Hamiltonian H_{br} . Using the explicit form of the Hamiltonian [14] (the leading nontrivial Hamiltonian of $\mathcal{O}(N^0)$), we can easily find the LC wavefunction (see Appendix B and Ref. [14]),

$$\Phi_{\alpha\beta}(\mathbf{k}) = \frac{C_\pi \lambda}{(2\pi)^3} \frac{M}{m} \frac{\{(ik_\perp^i \sigma^i + M)\sigma^3\}_{\alpha\beta}}{(k_\perp^2 + M^2)/m_\pi^2 - x(1-x)}, \quad (4.10)$$

where $\mathbf{k} = (xP^+, k_\perp^i)$ and C_π is just a normalization factor. The Pauli matrix σ^3 comes from γ_5 (see our definition (2.5)). The pion mass m_π is obtained from a normalization condition. If we use the “extended parity-invariant cutoff” [26,13]

$$\frac{k_\perp^2 + M^2}{x} + \frac{k_\perp^2 + M^2}{1-x} < 2\Lambda^2, \quad (4.11)$$

the pion mass for a small bare fermion mass m is estimated as

$$m_\pi^2 = \frac{NZ_\pi}{\lambda M} m + \mathcal{O}(m^2), \quad (4.12)$$

where Z_π is a cutoff dependent factor:

$$Z_\pi^{-1} = \frac{N}{8\pi^2} \left[\ln \left(\frac{1+\beta}{1-\beta} \right) - 2\beta \right], \quad \beta = \sqrt{1 - 2M^2/\Lambda^2}.$$

Obviously m_π vanishes in the chiral limit $m \rightarrow 0$, and this state is indeed identified with the NG boson. It should be noted that $m_\pi = 0$ is realized even though the pion is described by the lowest Fock component only. This is due to the exact cancellation of the kinetic energy and the potential energy (see Appendix B). Also, in the chiral limit, the LC wavefunction becomes independent of x , which implies that the pionic state looks like a point particle in the longitudinal direction. This agrees with the results in two-dimensional QCD.

¹⁰This is also justified by the $1/N$ expansion.

Similarly, we solved the LF bound state equation (4.7) for the scalar meson state σ . [14] In the chiral limit, the mass of σ is given by twice the dynamical fermion mass $m_\sigma = 2M$, and the LC wavefunction turns out to have a very narrow peak at $x = 1/2$. Thus, the scalar meson can be understood as the usual constituent state.

Now that we have the pion LC wavefunction, we can explicitly compute various physical quantities and relations. In addition to the form factor [32] and the distribution amplitude, [26] which are intimately related to the LC wavefunction, we can also calculate the pion decay constant f_π and can check the PCAC relation. The decay constant f_π is defined by

$$iP^\mu f_\pi = \langle 0 | J_5^\mu(0) | \pi; \mathbf{P} \rangle, \quad (4.13)$$

where we use the current (2.11) as mentioned above. The result is $f_\pi = 2M Z_\pi^{-1/2} + \mathcal{O}(N^0)$. Together with the pion mass (4.12), we verify the Gell-Mann, Oakes and Renner (GOR) relation,

$$m_\pi^2 f_\pi^2 = -4m \langle 0 | \bar{\Psi} \Psi | 0 \rangle. \quad (4.14)$$

The PCAC relation can also be checked by using the state $|\pi; \mathbf{P}\rangle$. Introducing the pseudoscalar field $\pi(x) = Z_\pi^{-1/2}(\lambda/N)\bar{\Psi}i\gamma_5\Psi(x)$ normalized as $\langle 0 | \pi(0) | \pi; \mathbf{P} \rangle = 1$, we arrive at the PCAC relation

$$\begin{aligned} \partial_\mu J_5^\mu &= 2m\bar{\Psi}i\gamma_5\Psi \\ &= m_\pi^2 f_\pi \pi(x), \end{aligned} \quad (4.15)$$

where we have used the usual current (2.11) again.

To this point, we have treated the broken phase Hamiltonian H_{br} to solve the LF bound state equation (4.7). What happens if we chose the symmetric phase Hamiltonian but with large coupling, $\lambda > \lambda_{\text{cr}}$? Recall that the gap equation has a symmetric solution $M \rightarrow 0$ ($m \rightarrow 0$) even for $\lambda > \lambda_{\text{cr}}$. In this case, if we solve the bound state equation, we have a *negative mass square*, which is not physically acceptable. [14] We can verify that the mass square is positive only for $\lambda < \lambda_{\text{cr}}$, where we have only one symmetric solution. Hence, the emergence of such tachyonic states should be a criterion for eliminating unphysical Hamiltonians.

D. Modified chiral transformation

Recall the discussion about the order parameters in Sect. IIB. There, we argued that the vacuum expectation value of the fermion bilinear operator $\langle \bar{\Psi} \Psi \rangle$ is not exactly an order parameter in the usual sense. This is because the usual treatments involving order parameters are based on the chiral transformation law (2.21), but if this law held in the LF formalism, it would always give $\langle \bar{\Psi} \Psi \rangle = 0$, due to $Q_5^{\text{LF}} |0\rangle = 0$. Since the invariance of the vacuum under the LF chiral transformation is a kinematical fact, and thus independent of the symmetry, the above observation is serious for the description of the broken phase. However, this can be resolved again by the broken phase solution of the characteristic constraints. Let us go back to the discussion in Sect. IVA. There, we argued that if we select the nonzero chiral condensate of the gap equation (3.24) in the NJL model, the fermionic field becomes a massive free fermion. This implies that *the operator structure of the dependent variables changes according to the phases*. In the chiral Yukawa model, the operator structure of the longitudinal zero modes, and subsequently of the bad spinor component in the broken phase, should be different from those of the symmetric phase. Therefore, the chiral transformation of the dependent variables also becomes modified. This can be easily understood from the fact that even the free fermion field transforms unusually if it has a nonzero mass:

$$[Q_5^{\text{LF}}, \Psi_M] = \gamma_5 \Psi_M - \sqrt{2}M\gamma_5 \frac{1}{i\partial_-} \psi, \quad (4.16)$$

where Ψ_M is the massive free fermion operator given by Eq. (3.17). Similarly, in the NJL model, the fermion field acquires a dynamical mass in the broken phase, and its transformation is necessarily changed. This “modified chiral transformation” in the broken phase propagates into the relation (2.21). Indeed, explicit calculation in the chiral limit of the NJL model [14] (and also of the chiral Yukawa model [13]) yields the modified transformation law

$$[Q_5^{\text{LF}}, \bar{\Psi}i\gamma_5\Psi(x)] = -2i\bar{\Psi}\Psi(x) + M\Delta(x), \quad (4.17)$$

where the extra term $M\Delta(x)$ is a function of the fermion operator. This unusual chiral transformation, however, is consistent with $Q_5^{\text{LF}} |0\rangle = 0$, because $-2i\langle 0 | \bar{\Psi}\Psi(x) | 0 \rangle + M\langle 0 | \Delta(x) | 0 \rangle = 0$. Conversely, as long as the vacuum

is invariant under the LF chiral transformation, the ordinary transformation law (2.21) must be modified as in Eq. (4.17). Note also that Eq. (4.17) implies that if we set $M = 0$ (the symmetric phase), the usual transformation law is recovered.

The same arguments hold for the sigma model [20] (and also for the chiral Yukawa model [13]). The transformation law of the scalar zero mode $\pi_0(x_\perp)$ is given by

$$[Q_5^{\text{LF}}, \pi_0(x_\perp)] = -2i\sigma_0(x_\perp) + \Delta\pi_0(x_\perp), \quad (4.18)$$

where $\Delta\pi_0$ is an extra term that does not appear in the usual transformation law.

All the operators including dependent variables should be transformed unusually in the broken phase under the LF chiral transformation. This is one of the inevitable consequences of the coexistence of the symmetry breaking and the trivial vacuum. Since we cannot compute the transformation of the dependent variables until solving the constraint equations, the extra terms inevitably become model dependent. This prevents us from giving generic discussion about the broken phase in the LF formalism. For example, it is very difficult to prove the NG theorem in general. However, we can say that the vacuum expectation value of the model-dependent extra term should always coincide with that of the order parameters to ensure trivial relations such as $\langle 0 | [Q_5^{\text{LF}}, \bar{\Psi}i\gamma_5\Psi(x)] | 0 \rangle = 0$ and $\langle 0 | [Q_5^{\text{LF}}, \pi_0(x_\perp)] | 0 \rangle = 0$.

E. Nonconservation of the LF chiral charge

The same arguments as those given in Sect. IVD hold for the Hamiltonian. This was already seen partly in Sect. IVB. The new concept of multiple Hamiltonians is a direct consequence of the fact that the Hamiltonian contains dependent variables. It is natural to expect that the chiral transformation of the Hamiltonian is also modified in the broken phase. The Hamiltonian is not invariant under the LF chiral transformation in the broken phase. In fact, in the NJL model, [14] we find the commutator $[Q_5^{\text{LF}}, H]$ is really nonzero in the broken phase, while zero in the symmetric phase:

$$[Q_5^{\text{LF}}, H_{\text{br}}] = M\Delta H_{\text{br}} \neq 0, \quad (4.19)$$

$$[Q_5^{\text{LF}}, H_{\text{sym}}] = 0. \quad (4.20)$$

In other words, the null-plane chiral charge Q_5^{LF} is not conserved in the broken phase. This has been pointed out by several people as a characteristic feature of the chiral symmetry breaking on the LF. [34,20]

The result (4.19) should be consistent with the current divergence relation Eq. (2.17). Integrating it over space, we have

$$\partial_+ Q_5^{\text{LF}} = \frac{1}{i} [Q_5^{\text{LF}}, H_{\text{br}}] = 2m \int dx^- d^2x_\perp \bar{\Psi}i\gamma_5\Psi. \quad (4.21)$$

Therefore if the LF chiral charge is not conserved in the chiral limit, the integral on the right-hand side must exhibit the singular behavior

$$\int dx^- d^2x_\perp \bar{\Psi}i\gamma_5\Psi \propto \frac{1}{m}. \quad (4.22)$$

By using the solution of the fermionic constraint, we can directly verify this singular behavior and the extra term in Eq. (4.19).

The importance of such singular behavior for making the NG boson meaningful has been stressed by Tsujimaru et al. in the context of scalar theories. [20] Assuming the PCAC relation, they showed that the zero mode of the NG boson has a singularity $\sim m_{\text{NG}}^{-2}$, where m_{NG} is an explicit symmetry-breaking mass. If we set $m = 0$ ($m_{\text{NG}} = 0$) from the beginning, the NG boson decouples from the system, and the NG phase is not realized. In our case, due to the GOR relation (4.14) and the PCAC relation (4.15), we can rewrite Eq. (4.22) as

$$\int dx^- d^2x_\perp \pi_0(x) \propto \frac{1}{m_\pi^2}, \quad (4.23)$$

which is the same form as that discussed in Ref. [20].

V. COUPLING TO GAUGE FIELDS

To this point, we have discussed scalar and fermionic fields. What occurs if we couple gauge fields to them? As we see below, this addition has a drastic effect, and the problem becomes greatly complicated. We discuss here the simplest example of the Abelian fields and give a brief comment on the non-Abelian case.

Using the complex scalar field $\phi = (\sigma + i\pi)/\sqrt{2}$, the $U(1)$ gauged sigma model can be equivalently rewritten as the Abelian Higgs model,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - V(\phi), \quad V(\phi) = \frac{\lambda}{4}(\phi^*\phi)^2 - \frac{\mu^2}{2}\phi^*\phi, \quad (5.1)$$

where $D_\mu = \partial_\mu - ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This model exhibits the Higgs mechanism, where the global $U(1)$ symmetry (which corresponds to the chiral symmetry in the sigma model) breaks down spontaneously and the photon acquires a nonzero mass. No NG boson appears because it is absorbed into the photon.

In the following, we consider 1+1 dimensions for simplicity, but it is straightforward to generalize our discussion to 3+1 dimensions. [16] As we have scalar fields, we use the DLCQ method. Periodic boundary conditions are imposed on the gauge fields as well as the scalar field. Thus, the zero modes we treat are a scalar zero mode ϕ_0 and a gauge field zero mode $\overset{\circ}{A}_- = \int_{-L}^L dx^- A_-(x)/2L$. In the Abelian Higgs model, the canonical structure of the zero mode is different from that of the sigma model. [16,17] To see this, let us consider the Euler-Lagrange equation of the Higgs field,

$$2D_- D_+ \phi - ie\phi \Pi^- + \frac{\partial V}{\partial \phi^*} = 0, \quad (5.2)$$

where $\Pi^- = F_{+-} = \partial_+ A_- - \partial_- A_+$ is the momentum conjugate to A_- . The covariant derivative D_- can be replaced by a normal derivative ∂_- if we introduce the (LF) spatial Wilson line

$$W(x^+, x^-) \equiv \exp \left\{ ie \int_{-L}^{x^-} dy^- A_-(x^+, y^-) \right\}. \quad (5.3)$$

Equation (5.2) can then equivalently be rewritten as

$$\partial_- [2W^{-1}D_+ \phi] = W^{-1} \left[ie\phi \Pi^- - \frac{\partial V}{\partial \phi^*} \right] \equiv \omega. \quad (5.4)$$

Unlike the case for the previous scalar model (see Sect. IIIA.), the integral of the left-hand side of this equation does not necessarily vanish, because the Wilson line is not periodic in general. This means that *only if the Wilson line is periodic*, i.e. $W(-L) = W(L)$, does the x^- integration of Eq. (5.4) generate a constraint analogous to the usual zero-mode constraint. If we employ the LC axial gauge, the unfixed gauge degree of freedom turns out to be $\overset{\circ}{A}_-$ only. The Wilson line is periodic when $\overset{\circ}{A}_-$ takes the discrete values $\overset{\circ}{A}_- = \pi N/eL$, $N \in \mathbf{Z}$. These discrete points correspond to the zero modes of the covariant derivative $D_- = \partial_- - ie\overset{\circ}{A}_-$. (Recall that the usual zero-mode constraint is related to the zero mode of the derivative operator ∂_- .) Gauge fields with different N are equivalent, because they are related by the large gauge transformation, which is associated with the nontrivial homotopy group $\pi_1(S^1) = \mathbf{Z}$. Therefore the canonical structure of the model is summarized as follows:

(A) Periodic Wilson line: $W(-L) = W(L) \iff \overset{\circ}{A}_- = \pi N/eL$.

Here there is the extra constraint $\int_{-L}^L dx^- \omega = 0$. One mode in the scalar field is a constrained variable.

(B) Non-periodic Wilson line: $W(-L) \neq W(L) \iff \overset{\circ}{A}_- \neq \pi N/eL$.

Here there is no extra constraint, and all the modes of the scalar field are dynamical. The gauge field zero mode $\overset{\circ}{A}_-$ is also dynamical.

For case (A), the same arguments as those given in the case of the sigma model are applicable. We can solve the constraint, and the nonperturbative (tree level) solution gives a nonzero vacuum expectation value of ϕ . However, this

is an exceptional point in the phase space, and we have to treat case (B) in general. In this case, we do not have the zero-mode constraint, but there is no problem. Our situation is rather similar to that for the usual ET calculation. We only need to evaluate the vacuum energy and find the true vacuum, as usual. It is easy to find classical configurations that minimize the LF energy P^- . The LF energy becomes zero if and only if the field configuration is given by

$$\phi = \frac{\mu}{\sqrt{\lambda}} e^{i\pi N \frac{x^-}{L}}, \quad \overset{\circ}{A}_- = \pi N/eL. \quad (N \in \mathbf{Z}) \quad (5.5)$$

This configuration gives a nonzero vacuum expectation value to the scalar field. Note that Eq. (5.5) should be classified as case (A). Indeed, the scalar field ϕ above is related to the zero-mode solution $\phi_0 = \mu/\sqrt{\lambda}$ via the large gauge transformation. We can formulate the quantum problem in case (B) by expanding the fields around this configuration. [17] Since the zero modes are all dynamical, the true vacuum is not the trivial Fock vacuum, but should be determined as the lowest energy eigenstate of the Schrödinger equation within the zero-mode sector.¹¹

Hence, the canonical structure in the zero-mode sector is not so simple, due to the presence of the gauge field. Depending on the periodicity of the Wilson line, we have to consider two cases separately, but it appears that these two cases are connected, and we can describe the symmetry breaking (the Higgs mechanism) in both cases.

The same structure is observed in non-Abelian gauge theories. For example, in the 3+1 dimensional $SU(2)$ Yang-Mills theory, [35] at the points where the gauge field zero mode takes the values $\overset{\circ}{A}_-^a = \delta^{a3} n\pi/gL$ (corresponding to case (A) above), there exist two constraint relations,

$$\int_{-L}^L dx^- e^{\frac{i\pi n}{L}x^-} (G^1 + iG^2) = 0, \quad (5.6)$$

where $G^a = (D_i F_{-i})^a$ or $G^a = (D_i F_{+-} + D_j F_{ji})^a$ for $a = 1, 2$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$. Noting that these originate from the zero eigenvalue problem of the covariant derivative D_- , we can easily imagine that the same kind of constraint relations will be present in any gauge theory, as well as in QCD. In Ref. [35], quantum theory in the case with the constraints (5.6) was not developed due to its complexity. The physical consequences of these constraints are still not known. As we saw in the Abelian Higgs model, however, the zero-mode constraint in case (A) has significant meaning, though it appears as a special case. Therefore it is interesting to investigate the consequence of the constraints (5.6) and to understand the zero-mode structure of the gauge theories.

VI. CONCLUSIONS

We have studied how to describe chiral symmetry breaking on the LF. Throughout the analyses, the most important point was to clarify the role of the characteristic constraint equations which are present only in the LF formalism. Many problems are resolved by carefully treating the constraint equations and their solutions. We have given clear answers to the three questions posed in the Introduction. Let us recapitulate them below as our conclusions.

- (i) The chiral transformation defined on the LF is different in general from the ordinary chiral transformation. This is because half of the fermion degrees of freedom (the bad component) are not independent, and we define the chiral transformation only on the independent degrees of freedom (the good component). The difference is evident in the free fermion theory, where the LF chiral symmetry is exact, even with the mass term. Nevertheless, as far as the models we studied in the present paper are concerned, such extra symmetry does not exist. This was explicitly checked by solving the constraint equations classically. If and only if we do not have a bare mass term, the LF chiral transformation is the symmetry of the system, and it is equivalent to the ordinary chiral symmetry. (Sects. IIB and III.)
- (ii) We have seen in several models that the zero-mode constraint and the fermionic constraint play the same role. The gap equations result from these constraint equations. This is natural, because the constraints were originally a part of the Euler-Lagrange equations and should carry the information concerning the dynamics. However,

¹¹Furthermore, we can explicitly construct the θ vacua by imposing on the vacuum invariance under the large gauge transformation. [17]

technically this is not evident, and we have to be careful about the infrared divergence of the longitudinal momentum integration. This is related to the fundamental problem in the LF formalism, the loss of the correct mass dependence from two-point functions. Without the proper treatment of the IR divergence, we cannot obtain meaningful gap equations. (Sects. III and IVA.)

- (iii) As a result of the coexistence of a nonzero chiral condensate and the trivial Fock vacuum, many unusual consequences follows. Most of them can be properly understood if we carefully treat the characteristic constraint equations.
 - (iii-a) Even if we select a nontrivial solution to the gap equation, the vacuum still remains the Fock vacuum. In general, there are several independent solutions to the gap equation, which implies several solutions to the constraint equations. Consequently, we obtain multiple Hamiltonians, but with one trivial Fock vacuum. Multiple vacua in the ET formalism corresponds to multiple Hamiltonians in the LF formalism. Effects of the usual nontrivial vacuum physics are converted into the Hamiltonian in the LF formalism. (Sect. IVB.)
 - (iii-b) Since we have a trivial vacuum, even the NG boson can be described by a few constituents. In the NJL model, we explicitly constructed the pionic state and obtained the LC wavefunction and the mass. Using the LC wavefunction, we can directly calculate various physical quantities (e.g., the pion decay constant) and relations (e.g., the GOR and PCAC relations). (Sect. IVC.)
 - (iii-c) Since the null-plane charge always annihilates the vacuum, it appears that a nonzero chiral condensate is inconsistent with the chiral transformation law of $\bar{\Psi}i\gamma_5\Psi$ (see Eq. (2.21)). However, in the broken phase, the chiral transformation of the dependent variables is necessarily changed, and, subsequently, the transformation law for $\bar{\Psi}i\gamma_5\Psi$ is also modified, so that there is no inconsistency in the equation (Eq. (4.17)). (Sect. IVD.)
 - (iii-d) The Hamiltonian in the broken phase does not commute with the LF chiral charge. In other words, the LF chiral charge is not conserved in the broken phase. This is essentially because the Hamiltonian contains dependent variables and they are transformed unusually in the broken phase under the LF chiral transformation. This is also related to the singular behavior of the spatial integration of the NG field, which appears in the limit of small explicit symmetry breaking mass. (Sect. IVE.)

Therefore, as far as we are concerned with scalar and fermionic systems, the dynamical chiral symmetry breaking can be satisfactorily described by the LF formalism.

The next problem to be addressed, of course, regards the effects of gauge fields. In the presence of the gauge fields, the canonical structure of the zero-mode sector becomes greatly complicated. In particular, the scalar and gauge-field zero modes remain dynamical in almost all of phase space. The physics of the zero mode is still important for describing symmetry breaking, and we indeed demonstrated this in the Abelian Higgs model (Sect. V). For chiral symmetry breaking in QCD, we need to further understand the physical implications of the gauge field zero modes.

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APPENDIX A: CONVENTIONS

We follow the Kogut-Soper convention. [2] First of all, the LF coordinates are defined as

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \quad x_\perp^i = x^i, \quad (i = 1, 2) \quad (\text{A1})$$

where we treat x^+ as “time”. The spatial coordinates x^- and x_\perp are called the longitudinal and transverse directions, respectively. Therefore, our metric is

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{A2})$$

for $\mu, \nu = (+, -, 1, 2)$. The inner product between two four vectors is given by

$$\begin{aligned} p \cdot x &= p_+ x^+ + p_- x^- + p_\perp i x_\perp^i \\ &= p_- x^+ + p^+ x^- - p_\perp^i x_\perp^i. \end{aligned} \quad (\text{A3})$$

From this, we can find that $p^- = (p^0 - p^3)/\sqrt{2}$ is the LF energy and $p^+ = (p^0 + p^3)/\sqrt{2}$ is the longitudinal momentum. Derivatives in terms of x^\pm are defined by $\partial_\pm = \partial/\partial x^\pm$. The inverse of $i\partial_-$ is defined by

$$\frac{1}{i\partial_-} f(x^-) = \int_{-\infty}^{\infty} dy^- \frac{\epsilon(x^- - y^-)}{2i} f(y^-), \quad (\text{A4})$$

where $\epsilon(x)$ is a sign function. The operator ∂^2 is given by $\partial^2 = 2\partial_+\partial_- - \partial_\perp^2$.

It is useful to introduce projection operators Λ_\pm defined by

$$\Lambda_\pm = \frac{1}{2} \gamma^\mp \gamma^\pm = \frac{1}{\sqrt{2}} \gamma^0 \gamma^\pm. \quad (\text{A5})$$

Indeed Λ_\pm satisfy the projection properties $\Lambda_\pm^2 = \Lambda_\pm$, $\Lambda_+ + \Lambda_- = 1$, etc. Splitting the fermion field by use of the projectors as

$$\Psi = \psi_+ + \psi_-, \quad \psi_\pm \equiv \Lambda_\pm \Psi, \quad (\text{A6})$$

we find that for any fermion on the LF, the ψ_- component is a dependent degree of freedom. ψ_+ and ψ_- are called the “good component” and the “bad component”, respectively.

As is commented in the text, for practical calculation, we use the two-component representation for the gamma matrices. The two-component representation is characterized by a specific form of the projectors (2.2). Then the projected fermions ψ_\pm have only two components. There are many possibilities that realize Eq. (2.2). For example, the specific representation

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix} \quad (\text{A7})$$

is used in Ref. [36]. In this paper, however, we choose the representation given in Eq. (2.4). This representation allows us to easily extract results of the 1+1 dimensions from those of the 3+1 dimensions.

APPENDIX B: LC WAVEFUNCTION OF A PION IN THE NJL MODEL

Here we discuss further the LC wavefunction of a pion in the LF NJL model. First, using the explicit form of the Hamiltonian (Eqs. (3.29)–(3.32) in Ref. [14]), we can find the spinor structure of a pseudoscalar state should be

$$\Phi_{\alpha\beta}(\mathbf{k}) = \{(ik_\perp^i \sigma^i + M)\sigma^3\}_{\alpha\beta} \phi_\pi(x, k_\perp^i), \quad (\text{B1})$$

where $\mathbf{k} = (xP^+, k_\perp^i)$. The Pauli matrix σ^3 which comes from γ_5 ensures that this is a pseudoscalar state. The spinor independent part $\phi_\pi(x, k_\perp^i)$ satisfies the integral equation

$$m_\pi^2 \phi_\pi(x, k_\perp^i) = \frac{k_\perp^2 + M^2}{x(1-x)} \phi_\pi(x, k_\perp^i) - \frac{\lambda\alpha}{(2\pi)^3} \frac{1}{x(1-x)} \int_0^1 dy \int d^2 \ell_\perp \frac{\ell_\perp^2 + M^2}{y(1-y)} \phi_\pi(y, \ell_\perp^i), \quad (\text{B2})$$

where α is just a numerical factor: $\alpha^{-1} = m/M + 2\lambda/(2\pi)^3 \cdot \int d^2 q_\perp \int_0^1 dx/x$. Its solution is easily found as

$$\phi_\pi(x, k_\perp^i) = C_\pi \frac{\lambda}{(2\pi)^3} \frac{M}{m} \frac{1}{(k_\perp^2 + M^2)/m_\pi^2 - x(1-x)}, \quad (\text{B3})$$

where C_π is a normalization constant:

$$C_\pi = \int_0^1 dx \int d^2 k_\perp \phi_\pi(x, k_\perp^i). \quad (\text{B4})$$

Combining Eqs. (B1) and (B3), we obtain Eq. (4.10). Equation (B4) also leads to an equation for m_π (Actually C_π is determined by the normalization condition Eq. (4.9))

$$\frac{1}{\lambda} = \frac{M}{m} \int_0^1 dx \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{m_\pi^2}{k_\perp^2 + M^2 - m_\pi^2 x(1-x)}. \quad (\text{B5})$$

This nonlinear equation can be analytically solved for small bare mass case. As is shown in the text, if we introduces the cutoff (4.11), we find Eq. (4.12).

In Eq. (B2), the first term corresponds to the kinetic energy part of the fermion and antifermion, with constituent mass M , and the second term corresponds to the potential energy part. The constituent picture for the massless pion is realized by an exact cancellation of the kinetic and potential terms. Note that the kinetic term is just a summation of the kinetic energies of the fermion and antifermion with mass M .

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